A STUDY OF VIBRATIONS OF ROTATING LARGE ASPECT RATIO BLADES

By

BHUPINDER NATH

AE
1976
M
NAT
STU

TH
AE / 1976 / m
B 4695

Thesis
629.132362
B 4693

Department of Aeronautical Engineering

INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

July, 1976

A STUDY OF VIBRATIONS OF ROTATING LARGE ASPECT RATIO BLADES

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of
MASTER OF TECHNOLOGY

By
BHUPINDER NATH

to the

Department of Aeronautical Engineering
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR

July, 1976

CENTRAL LIBRARY
Acc. No. A 47186
14 OCT 1976

POST GRADUATE OFFICE
Submitted on 14.7.76

CERTIFICATE

Certified that the present investigation ''A Study of Vibrations of Rotating Large Aspect Ratio Blades'' by Bhupinder Nath has been carried out under my supervision and has not been submitted elsewhere for the award of a degree.

July - 1976

N. G. R. Iyengar

[N.G.R.Iyenger]
Assistant Professor
Department of Aeronautical Engineering
Indian Institute of Technology,Kanpur.

POST GRADUATE OFFICE
This thesis has been approved
for the award of the Degree of
Master of Technology (M.Tech.)
in accordance with the
regulations of the Indian
Institute of Technology Kanpur
Dated. 21/8/76

ACKNOWLEDGEMENT

The author wishes to record his gratitude to his thesis advisor, Dr. N.G.R.Iyenger for suggesting this interesting problem and for his constant encouragement and guidance throughout the course of this work.

The author is also thankful for the suggestions of Dr.A.K.Mallik during the course of this investigation.

The author is grateful to the Aeronautics, R and D Board, Ministry of Defence, Government of India who supported this work by a grant under Project No.5/AE/12.

The author wishes to convey his thanks to his friends, Mr. Sutesh Kumar, Mr.P.L.N.Murthy,Mr.Kuldip Singh and Mr. Kultar Singh for their help and suggestions to finish this work successfully.

Thanks are also due to Mr. A.K.Bhatia for cutting stencils and Mr. A.N.Upadhyay for cyclostyling them and to Mr. Ramesh Kumar for his fine drawings.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

NOTATION AND SYMBOLS

$|A(x)|$ = area of cross-section

$|B(x)|$ = state matrix for pure bending mode

$|C(x)|$ = state matrix for combined bending and torsion mode

$D(x)$ = depth variation

E = Young's modulus

e = distance between centre of mass and shear centre

G = modulus of rigidity

I = moment of inertia

$|I|$ = identity matrix

$I_p(x)$ = polar moment of inertia

$i_y^2(x)$ = radius of gyration

J = mass moment of inertia about centre of mass

K = rolling stiffness factor

K_s = shape factor

L = length of the beam

$M(x)$ = bending moment

$M_T(x)$ = torsional moment

NS = free end

o = fixed end

r = radius of the disk

$|T|$ = overall transfer matrix

$|U|$ = local transfer matrix

$V(x)$ = shear force

$v(x)$ = deflection at the centre of mass

$W(x)$ = width variation

$w(x)$	=	dynamic deflection of the beam
x	=	distance measured along the length of the blade from the fixed end
$\|z(x)\|$	=	state vector
θ	=	stagger angle in radians
$\beta(x)$	=	torsional deflection
γ	=	width taper parameter
δ	=	depth taper parameter
$\psi(x)$	=	bending slope
μ	=	mass per unit length
Ω	=	speed of rotation in RPM
ω	=	natural frequency in cps
λ	=	frequency parameter of a tapered beam
λ_o	=	fundamental frequency parameter of a uniform beam
λ/λ_o	=	frequency parameter ratio

ABSTRACT

In this investigation the transfer matrix approach is used to determine the natural frequencies and mode shapes of a rotating large aspect ratio asymmetric tapered turbine blade. Some of the predominant factors like shear deformation, rotary inertia, root flexibility and stagger angle which affect the natural frequencies of the blades are also studied.

Runge-Kutta method is used to solve the matrix of the variable co-effecients of system of first-order differential equations of motion for coupled bending-torsion and pure flexural modes.

Typical results are presented in the tabular as well as graphical form.

CHAPTER I

INTRODUCTION

1.1 Description of the Problem

Knowledge of the resonant characteristics of turbine and compressor blades is of utmost importance in insuring reliable long-life- turbine engines. The damage caused by blades failing due to vibratory fatigue can be catastrophic at worst and at the very least result in additional engine development costs due to redesign and repair. Comprehensive analytical and experimental methods for analyzing the vibrational characteristics of turbine blading are essential in the avoidance and prediction of turbine engine blade vibration problems.

With the large increase in the use of axial flow compressor and turbine, determination of natural frequencies and modes of vibration is of considerable importance in the design of blades used in turbomachines. It is known from earlier investigations that most of the blade failures occur due to fatigue at the blade roots, and it occurs when vibrations take place at or near resonant conditions. For the safe design, the designer has to make sure that the natural frequencies obtained by theoretical calculations are reasonably away from the forcing frequencies.

The difficulties facing the design of blading are

many. The magnitude of the problem lies in the fact that in some turbomachines there can be as many as a few thousand fixed and rotating blades of different characteristics and the failure of even one of them will force a shutdown. The major problem of design is a fair prediction of the natural frequencies of the blading at the earlier stages by theoretical calculations.

In determining the natural frequencies, the designer can idealize the large aspect ratio blade as a uniform cantilever beam and find the effects of the following primary important factors :

1. Rotary inertia and Shear deformation, 2. Rotation of the disk, 3. Taper of the blade cross-section, 4. Stagger angle of the blade, 5. Root flexibility, 6. Disk elasticity, 7. Disk radius, 8. Pretwist of the blade, 9. Asymmetry of the blade, 10. High temperature, 11. Aerodynamic excitation, 12. Aspect ratio of blade.

As turbomachine blades are often joined together,either at their tips—shrouding or at some intermediate location -- lacing. In addition to the above mentioned parameters few more parameters that affect the response of blades are as follows :

1. Shrouding, 2. Lacing wires, 3. Large amplitude, 4. Damping, 5. Material.

A good deal of theoretical and experimental work is reported in the literature which enables the designer to

calculate the natural frequencies, but it is still a long way to go before the last word can be said on this problem. As one cannot entirely depend upon the theoretical calculations, it becomes essential to make full-scale model tests on the prototype and study the vibration characteristics, i.e., the natural frequencies and their corresponding nodal patterns. But the experimental procedure is time-consuming and it is very expensive to conduct tests on a number of designs. Thus it becomes essential for the disigner to have theoretical knowledge as well as practical means for conducting model tests in order to predict the vibration characteristics of the blading at an earlier stage of design.

The most important modes of vibration from practical considerations, are known to be the first three flapwise bending modes, the first chordwise bending mode and the first two torsional modes. Depending on the characteristics, the turbine blading can execute either uncoupled bending or torsional vibrations or coupled bending torsion vibrations. Coupled bending torsion vibrations occur when the centre of flexure does not coincide with the centroid of the blade cross-section. When the blade is pretwisted, vibrations are further coupled between the two bending modes. The differential equations of motion for the case of a pretwisted blade with asymmetric cross-section are quite complex and closed form solution of such equations is difficult to obtain.

Also the natural frequencies determined are affected by shear deformations, rotary inertia, warping of the cross-section, root fixing and coriolis accelerations. In general the equations of motion will be six coupled partial differential equations coupled between the two bending deflactions, the two shearing deflections, the torsional deflection and the longitudinal deflection. Further the warping function of the blade cross-section will be obtained by a modified Poission equation, taking into account the dynamic conditions of the blade. Since the longitudinal natural frequency of the blade is generally very high and there is less possibility of exciting a longitudinal mode of the blade and hence this deflection may be ignored. Carnegie [1] considered the coriolis accelerations due to the inward displacement of the blade element and derived the nonlinear partial differential equations which are very complex, it would be a futile exercise in the present day to attempt a solution of them taking into account all the effects. When the blades are shrouded, the coupling between the blade and the shroud is to be further taken into account. Thus, theoretically it is an uphill task to determine the natural frequencies of an actual turbine blade with all the effects mentioned.

As stated above, the number of parameters which affect the response of the blades are quite large and is rather impossible to consider all these aspects at one time.

Furthermore, to evaluate their relative importance, it is easy to incorporate these parametersone by one.

In this investigation, transfer matrix approach is used to determine the natural frequencies and mode shapes of a rotating blade on the low temperature side of the turbine — idealized as a large aspect ratio cantilever beam tapered in both depth and width with different support flexibilities. The formulation of the problem is derived for pure bending and coupled bending torsion modes of vibration — including some of the predominant factors like shear deformation, rotary inertia, root flexibility, stagger angle etc.

Chapter II contains up- to-date literature survey on most of the parameters considered to be effective for affecting the response. Chapter III and IV cover the formulation of the problem for pure bending and coupled bending torsion modes of vibration respectively. Numerical results and conclusions are presented in Chapter V. Chapter VI deals with the recommendations for further study in this direction.

CHAPTER II

2.0 LITERATURE SURVEY

In the last twenty five years a large number of technical papers have appeared, describing the different approaches for the vibration analysis of turbomachinery blades. The general methods of solving the blade vibration problems are :

1. Exact method, 2. Rayleigh's method, 3. Rayleigh- Ritz method, 4. Ritz and Galerkin method, 5. Matrix methods, 6. Prohl method, 7. Numerical methods to determine the frequency equation, 8. Integral equation approach, 9. Finite difference method, 10. Finite element method, 11. Perturbation method, 12. Frequency response methods.

The classical approach of solving the differential equations of motion of a cantilever blade is discussed in most of the books on vibration. The solution is possible only under simplifying conditions. Matrix methods are considered by Duncan [2], Duncan and Collar [3], Dokumaci [4], Carnegie, Thomas and Dokumaci [5] and Scanlan and Rosenbaum [6]. Prohl [7] method is very successful for bending vibration problems of the blades.

Based on Holzer and Myklestad equations, Rao [8,9] developed numerical procedures to determine frequency equations. The integral equation approach is considered by Lee and Bishop [10], and White [11]. The perturbation method is used by Martin [12,13]. The frequency response methods are

described by Rubin [14,15], and Leckie and Pestel [16].

It is well knownthat the bending and torsional modes get coupled because of the cross-sectional properties. In what follows, the techniques generally used for solving coupled and uncoupled systems are discussed along with the methods to incorporate the other parameters. For the sake of convenience this review of the analytical methods are subdivided into several sections, in such a manner, that in each section a different aspect of the problem is discussed.

2.1 BENDING VIBRATIONS

The natural frequencies for cantilever beams of uniform crosssection can be determined by classical methods, and are given by

$$\omega_n = \frac{a_n^2}{L^2} \sqrt{EI/\mu} \tag{2.1}$$

where EI is the flexural rigidity, μ the mass per unit length and L the length of the beam. The values of a_n for the first four modes are 1.875,4.694, 7.855 and 10.996.

For tapered beams it is more convenient to use numerical methods. The tabular method of Myklestad [17] is, probably, the most widely used. In this method the beam mass is lumped at a number of discrete stations along the beam. A discussion of different methods available for forming the lumped parameter systems is given by Minhinnick [18]. Leckie and Lindberg [19] have considered the effect

of lumped parameters on bending vibration.

2.2 TORSIONAL VIBRATIONS

The classical method for the analysis of torsional vibrations of circular sections result in a differential equation of motion of the second order. For cantilever rods the substitution of boundary conditions of zero twist at the fixed end and zero torque at the free end yields the expression for torsional frequencies in the form

$$\omega_n = \frac{n\pi}{2L}\sqrt{Cg/I_{c.g}} \qquad [\, n = 1,3,5,\ldots] \qquad \ldots\ldots [2.2]$$

where C is the torsional rigidity, L the length and $I_{c.g}$ the polar moment of inertia per unit length of the rod. Sometimes this expression is also used to give approximate frequencies for non-circular rods by using the value of C given by the Saint Venant's theory of torsion.

For an aerofoil section the torsional rigidity cannot be calculated easily and recourse is generally made to experimental and empirical methods. Jawson and Ponter [20] have suggested a method for calculating the torsional rigidity of any arbitrary cross-section, to an accuracy of one percent.

Scanlan and Rosenbaum [6] gave Rayleigh's method, the Rayleigh-Ritz method, and matrix iteration procedure for calculating torsional frequencies of cantilever blades. Walker [21] gave simple formulae for the fundamental torsional mode obtained by substituting the original system, by a torsion member without mass and having single moment of inertia.

Uncoupled torsional vibrations of uniform and nonuniform cantilever beams have been considered by Duncan [2], Houbolt and Anderson [22], and Wittmeyer [23,24]. Duncan and Collar [3], Pipes [25] and Thomson [26] gave matrix solutions for uncoupled torsional frequencies of cantilever beams. Vet [27] gave results of torsional vibration of rectangular crosssection cantilever beams, for general ratios of length to depth and width to depth, obtained by the use of the Rayleigh- Ritz method.

2.3 COUPLED BENDING-TORSION VIBRATIONS

Coupled bending torsion vibrations occur when the centre of flexure does not coincide with the centroid of the blade crosssection. Garland [28] considered the case of such a cantilever blade and used the Rayleigh-Ritz method to determine the frequencies and amplitude ratios. Isakson and Eisley [29] considered the case of a pretwisted beam vibrating in coupled bending-torsion and determined the frequencies using an extention of the Holzer-Myklestad method. Rao [30] developed a numerical procedure to determine the frequency equation of a turbine blade executing bending-torsion vibrations and studied the effects of coordinate distance between the centre of flexure and the centroid on the natural frequencies of a straight uniform blade with asymmetric aerofoil crosssection. Torsion introduces root fixing, i.e., at about (0.2 X Chord) of blade length above the root is rigid in torsion.

2.4 COUPLED BENDING-BENDING VIBRATIONS

If the blade has pretwist, uncoupled bending modes in flapwise and chordwise directions are not possible. Both the lateral deflections are always coupled and coupled bending-bending vibrations occur.

Dunholter [31] considered the static displacements and natural frequencies of pretwisted beams. White [32] employed Green's functions to derive the conditions of orthogonality for a nonuniform pretwisted blade executing bending-bending vibrations. Mendelson and Gendler [33] used a method of station functions to study the effect of pretwist on lateral vibrations of cantilever blades. Diprima and Handelman [34] solved the equation of motion of a pretwisted cantilever blade by the Rayleigh-Ritz principle, Targoff [35] considered the case of a pretwisted rotating beam executing lateral vibrations. Carnegie [36] considered the effects of pretwist on lateral vibrations of pretwisted blades. Dawson and Carnegie [37] determined the coupled natural frequencies of a pretwisted cantilever blade by a transformation method. Rao [38] used Galerkin method to determine the first five natural frequencies of a pretwisted tapered cantilever blade. Weidenhammer [39] used Variational principle to compute the lowest eigen frequencies of the coupled system. Using the finite difference method, Carnegie and Thomas [40,41] considered pretwisted tapered blading and showed good agreement between theoretical and experimental results.

The variation in the frequency of vibration of a uniform beam due to pretwist depends on the pretwist angle and the width-to-depth ratio of the cross-section.

The influence of taper onto an otherwise uniform square cross-section beam makes the principle flexural rigidities of the beam unequal except at the root. When such a tapered beam is pretwisted, coupling occurs between the bending motions in the two mutually prependicular planes. The degree of coupling is a function of pretwist angle and taper.

2.4.1 TORSIONAL VIBRATIONS

For a pretwisted blade of rectangular section the torsion and the bending modes remain uncoupled. The torsional stiffness, however, is very much increased due to pretwist, resulting in an increase in the torsional frequencies. The main cause of the increase in the torsional stiffness over and above that given by the Saint Venant's theory, is due to the inclination of the blade longitudinal fibres which become helical due to the initial twist.

Chen [42] considered the effect of pretwist on the torsional rigidities. Bogdanoff and Horner [43] gave the results for the influence of rotation on the first three natural frequencies in torsion of a uniform cantilever beam with different base setting angles and constant pretwist rates. Brody and Targoff [44] considered the torsional vibration of a

thin pretwisted beam. Carnegie [45] considered an additional effect due to torsion on torsional vibration of pretwisted cantilever blades. Rao [46] considered the effects of fiber bending on the torsional vibration of pretwisted cantilever blades.

2.4.2 COUPLED BENDING-BENDING-TORSION VIBRATIONS

In a blade of aerofoil cross-section, apart from the coupling between the bending modes, produced by pretwist, the torsional vibrations are also coupled. Houbolt and Brooks [47] derived the differential equations of motion for pretwisted cantilever beams with asymmetrical aerofoil cross-section and suggested the Rayleigh-Ritz method for their solution. Carnegie, Dawson and Thomas [48] determined the natural frequencies and mode shapes for the first five modes of coupled bending-bending-torsion vibrations, using a method of finite differences. Dawson [49] has determined the natural frequencies of cantilever beams executing coupled bending-bending-torsion vibrations by Rayleigh-Ritz method. Carnegie and Dawson [50] used the method of finite differences and an analytical process in which the three coupled equations are transformed to an ordinary tenth order differential equation, to determine the natural frequencies and mode shapes for a straight blade of asymmetrical airfoil cross-section. Rao and Carnegie [51] used Galerkin method to determine the first five coupled frequencies of a straight uniform blade with asymmetric aerofoil cross-section.

2.5 EFFECTS OF SHEAR DEFORMATION AND ROTARY INERTIA

For stubby beams and where higher modes are required, the Bernoulli-Euler equation of motion for lateral vibration introduces considerable error. Rayleigh introduced the concept of rotary inertia, which was later extended by Timoshenko [52] who obtained an equation allowing for both rotary inertia and shear deformation by a conventional Newtonian approach. Carnegie [53] derived the differential equation of motion by the use of calculus of variations.

Sutherland and Goodman [54] solved the Timoshenko beam equation for the case of simply supported and cantilever beams. Chaplin [55] considered the effects of rotary inertia and shear deflection while determining the lateral natural frequencies of the blades of a 12- stage turbine. Hurty and Rubenstein [56] derived generalized mass and stiffness matrixes from the Kinetic and strain energy expressions and presented results for a uniform simply supported beam. Lee and Bishop [57] used integral equations for the solution of flexural vibrations of a wedge taking into account both the effects of rotary inertia and shear. Carnegie [58] studied the effects of shear deflection and rotary inertia for straight and pretwisted uniform cantilever beams. Dawson [59] used the Rayleigh-Ritz method and showed good agreement between his results and those of Reference [54]. Rao [60] used the Ritz method to determine the natural frequencies in lateral vibrations of tapered cantilever beams allowing for

shear and rotary inertia. Krupka and Baumanis [61] considered the bending-bending mode of a rotating tapered pretwisted turbomachine blade including the effects of shear and rotary inertia. Dawson, Ghosh and Carnegie [62] presented the modal curves of pretwisted blading allowing for rotary inertia and shear deflection obtained by Runge-Kutta process. Others who contributed to the work include Jacobson [63,64], Scanlan [65], Kruszewski [66], Fettics [67,68], Mindlin and Deresiewicz [69] , Cowper [70], Gaines and Volterra [71].

The frequency parameter decreases due to shear deflection and rotary inertia as the root width-to-length ratio of the beam increases, for any taper.

2.6 EFFECT OF TAPER

The mathematical solution for a straight uniform beam in simple bending was well known as early as 1736. Meyer [72], Ward [73], Nicholson [74], Wrinch [75], Akimasa [76], Conway [77], Watanabe [78], Nubuo [79], and Wang [80] gave solutions of lateral vibrations of cantilever beams with variable cross-section. Southwell [81] gave a graphical method for lateral vibration of nonuniform beams. Matrix methods using influence coefficients were given by Duncan and Collar [3], Thomson [26], Leckie and Lindberg [82], and Lindberg [83]. Martin [84] derived correction factors for the effect of taper on flexural frequencies by expressing the frequency as a double power series on two taper parameters of breadth and thickness. Taylor [85] solved the basic differential

equations of flapwise blade bending by omitting structural stiffness terms. Housner and Keightley [86] determined the natural frequencies and mode shapes of variable cross-section cantilever by means of a digital computer, using Prohl method. Frequency parameter ratios for the first five modes of tapered cantilever beams in lateral vibration are given by Carnegie, Dawson and Thomas [48], Rao [87] determined the uncoupled bending and torsional frequencies of tapered cantilever beams by using the methods of Galerkin. Rissone and Williams [88] studied the frequencies of tapered beams over a small range of depth ratios. Thomas [89] used the finite difference method to determine the bending frequencies of tapered cantilevers. Carnegie and Thomas [90] gave the natural frequencies of long tapered cantilevers using the finite difference method. Rao and Carnegie [91] determined the first three lateral frequencies of tapered beams by the use of the Ritz-Galerkin process.

Thomas [89] and Thomas and Carnegie [92] used the finite difference method to study the effect of taper on the torsional vibration of Cantilever blade. Rao [93] used the Galerkin method to consider the effect of thickness taper on the torsional vibration of cantilever beams with rectangular cross-section. Mabie and Rogers [94] considered doubly tapered cantilever beams in transverse vibrations.

The variation in width taper has little effect on the

decrease due to shear deflection and rotary inertia. The increase in depth taper increases the effect of shear deflection and rotary inertia, thus reducing the frequencies.

2.7 EFFECTS OF ROTATION

The natural frequencies of rotating turbomachinery blades are known to be significantly higher than those of the non-rotating blades. The design speed for a turbomachine is generally established by drawing a Campbell diagram [95] for each row of the blades. Essentially, this diagram is a plot of the variation of the first few natural frequencies of blades with the speed of rotation of the machine, with engine order lines superimposed on it. All points of intersection on the diagram represent possible resonance if appropriate excitations are present. For the mechanical design of the blades, therefore, the knowledge of the blade natural frequencies, at various speeds of rotation, is important.

Lo and Renbarger [96] derived the differential equation of motion of a cantilever blade mounted on a rotating disk with stagger and showed that the frequencies of a bar vibrating transverse to a plane inclined at an angle to the plane of rotation can be found by a simple transformation of the frequencies of a bar vibrating perpendicular to the plane of rotation. Lo [97] in his analysis simplified the nonlinear problem by assuming the blade to be rigid everywhere except at the root and presented the solution in a phase plane. Isakson and Eisley [98] considered a similar model to

determine the natural frequencies in coupled bending and torsion of twisted rotating blades. Boyce, Diprima and Handleman [99] used Rayleigh-Ritz and Southwell methods to determine the upper and lower bounds of natural frequencies of a turbine blade vibrating perpendicular to the plane of rotation. Yntema [100] making use of the bending mode of a nonrotating beam determined the first three modes of a rotating beam by the use of a Rayleigh energy approach. Boyce [101] used Rayleigh-Ritz and Southwell methods to determine the upper and lower bounds of natural frequencies and showed that the frequencies depend almost linearly on the hub radius for various rotational speeds. Carnegie [102] derived a theoretical expressions for the work done due to centrifugal effects for small vibrations of rotating cantilever beams and established an equation for the fundamental frequency of vibration by the use of Rayleigh's method. Schilhansl [103] investigated the stiffening effect of the centrifugal forces on the first mode frequency of a rotating cantilever blade by the use of successive approximation.

Carnegie, Stirling and Fleming [104] solved the differential equation of motion by the finite difference method to study the effects of rotation and stagger angle of the blade. Kissel[105] has given the modified Southwell factor for the effect of rotation. Rao [106] used the Galerkin method to study the effects of rotational speed, disk

radius and stagger angle of the blade and obtained a simple relation to determine the frequency of lateral vibrations. Rao [107] developed a numerical procedure to determine a polynomial frequency equation of a rotating cantilever beam. Rao and Carnegie [108,109] used the Ritz energy process and an averaging procedure to solve the differential equation derived by Carnegie [110] and obtained the nonlinear response of a cantilever blade. Ansari [111] applied Perturbation and Ritz energy methods to the problem of non-linear flexural vibration of a twisted, rotating cantilever blade of symmetrical cross-section vibrating in the plane of rotation. Pnueli [112] compared natural bending frequency to rotational frequency of rotating cantilever beam. Kundu [113] considered transverse vibration of an isotropic rotating, variable cross-section beam. Wauer [114] determined the eigenvalues of rotating turbine blade by means of Ritz method. Trompette and Lalanne [115] did the theoretical and experimental vibration analysis of rotating turbine blades, including large strains, root's influence, temperature effects using the finite element method. Whereas, Henry and Lalanne [116] performed the vibration analysis of rotating compressor blades. Ansari [117] presented an analysis for evaluation of non-linear modes of vibration of a pretwisted non-uniform cantilever blade of unsymmetrical cross-section mounted on the periphery of a rotating disk, including shear deformation, rotary inertia, cariolis effects, using the discrete system

formulation. Others who contributed to the work include Liebers [118,119], Love and Silberstein [120], Billington [121], Sutherland [122], Kobori [123], Plunkett[124], Liner [125] and Giansante [126].

2.8 EFFECTS OF ROOT FLEXIBILITY

In most of the theoretical investigations, the blade is assumed to be completely fixed at the root, in practice, however, the blade is rather loosely mounted on the rotor disk. The vibrations of the blade, therefore, continues to some extent into the mounting. This has an effect similar to that of increased blade length and results in a lower frequency than the one calculated on the fixed end assumption. The root flexibility, generally, makes the root a non-deflecting end, but not a non-rotating end. As a result, the slope at the root is not zero but is proportional to the moment acting at the root. The root flexibility varies with the speed of rotation, since the centrifugal forces tend to tighten the blade in the disk and increase the fixidity. Moreover, various types of root fixings are used in practice and, therefore, a general analytical treatment of this effect is rather difficult. Some investigations, however, have been carried out and are discussed here.

Niordson [127] has investigated the vibrations of turbine blades with loose hinge support in centrifugal force field. Hirsch [128] considered the effect of non-rigid support.

Traupel [129] has suggested an empirical method of taking into account the effect of root flexibility on the fundamental natural frequency in bending. He proposed the relation,

$$\omega_a = C_v \cdot \omega$$

where ω is the frequency calculated on the fixed end assumption and ω_a is the actual frequency. He has plotted the value of C_v against the ratio (λ) of the blade length to the radius of gyration of the root section, which is considered as a measure for the flexibility of the root. Baur [130] has suggested that the value of C_v given by Traupel [129] may also be used for the second flexural mode. For torsional vibrations, Baur [130] has suggested a similar relation, but has pointed out that the variation of the correction factor be plotted against the non-dimensional blade flexibility I_p/C, where I_p and C are the polar moment of inertia and the torsional stiffness at the root, respectively. Horway [131] considered hinged ends and Bogdanhoff [132] considered the effect of pinned ends on the natural frequencies of rotating blades.

The problem of pin-fixed blade is considered in more detail by Goatham and Smailes [133]. The blade is considered mounted on the disk through a loosely fitting pin, with a clearance. The pin tightens when the blade rotates, due to the centrifugal forces on the blade. Shaw [134] proposed an improved blade root design for the axial flow compressor and turbines.

2.9 EFFECTS OF DISK ELASTICITY

Jet engine rators are highly stressed. Considerable care must be taken for designing such systems to ensure that the parts do not fail during its life time.

If the disk on which the blades are mounted is considered absolutely rigid, each blade can be treated as a separate unit for vibration analysis. However, the disk has some elasticity, and hence the blades and the disk ought to be considered as an assembly. Energy from a blade can be transferred to the adjacent blades through the disk, which modifies the vibration characteristics of the blades.

Ellington and McCallion [135] have investigated the effect of elastic coupling through the disk rim, on the frequencies of vibration of the blades, by considering a simplified model. In this model, the blades are replaced by uniform beams fixed to the rim at their roots and vibrating in a plane parallel to the plane of the disk. The rim is considered as a uniform elastic ring, permitting no radial displacement at the roots of the blades. The mass of the rim is either neglected or considered as being concentrated at the root of the blades. The analysis has shown that if there are N identical blades mounted on an elastic rim, the system has N fundamental frequencies, N first overtones etc. However, if stiffness of the rim is large compared to that of the blade, as is generally the case, these N frequencies are very near to the corresponding frequency of

the individual blade. Thus instead of individual natural frequencies, there are bands of frequencies-- width of band decreasing with the increasing stiffness of the disk. This explains to some extent the scatter in the natural frequencies usually observed in vibration tests.

The coupling effects due to the blade ring, for the cases when either all blades have identical frequencies or alternate blades have identical frequencies, have been investigated by Sohngen [136]. Fillipov [137] has considered the problem of the tangential vibration of the blades together with the torsional vibrations of the disk, when all the rotating blades vibrate in phase. A simplified model for a row of blades mounted on a flexible disk is suggested by Wagner [138]. By using a model of blade-disk assembly more or less similar to one suggested by Wagner [138], Dye [139] has studied the effects on the stress level of the blades due to different factors. The factors taken are :
(a) the blades of a particular stage having slightly varying natural frequency and, (b) the transmission of forces from a blade to its neighbour through the flexible disk.

Shen [140] developed a mathematical formulation for the analysis of transient and steady state flexible rotor dynamics, including the general non-axisymmetric and non-synchronous rotor motion which may result from the included in -phase and out of phase stiffness and damping functions at all bearing and rotor stations. Other rotor dynamic parameter

considered are the rotor masses and mass moment of inertia and their eccentricities and misalignments. The effects of rotor drive and disipative torque and the interaction between torsional and transverse motion are also included.

Lemke and Trumpler [141] presented an analytical investigation of the effect of coupling characteristics on the response of axially coupled sets of turborotors. The approach used is transfer matrix technique which utilizes a generalized machine element matrix and coupling fixity matrix. Disk inertia, gyroscopic effects, and shaft as well as coupling orthotropy are included in this study.

Dopkin [142] developed a general analysis to predict the natural frequencies of vibration of a rotating shaft on linear, undamped bearings ; including flexible disks. The relative merits of several possible analytical methods are examined, and the transfer matrix technique is chosen to solve the system. The shaft and disks are described by a number of constant thickness segments and lumped masses. A fortran IV computer program is presented which solves the system equation by successive trials. Results of the analysis are found to agree very well with the results obtained by others for either flexible disks or flexible rotors with stiff disks.

A limited parametric study of six different rotor geometries with a range of disk flexibilities is presented. Some of the conclusions resulting from this study are :

(1) Disk flexibility always reduces the rotor natural frequency, sometimes to a few percent of the rigid disk value.

(2) When disk frequency is greater than four times rotor frequency, disk flexibility has a negligible effect.

(3) When the disk and rotor have the same frequency, the combined frequency is in the neighbourhood of 2/3rd as great.

(4) When rotating speed is greater than 1/2 of the natural frequency, a lower limit on natural frequency can be obtained with a rigid disk analysis by setting the disk polar and diametral moments of inertia equal to zero.

2.10 EFFECTS OF PRETWIST

If the blade has pretwist, uncoupled bending modes in flapwise and chordwise directions are not possible. Both the lateral deflections are always coupled and coupled bending-bending vibration occur.

Young [143] presented the effect of twist on blade vibrations. Thomas and Dokumaci [144] used Finite element method for pretwisted blade vibration. From the results obtained by various investigators, certain general conclusions can be drawn regarding the effect of pretwist on the frequencies of cantilever beams of rectangular cross-section. The conclusions are valid for the magnitudes of pretwist usually encountered in turbomachinery blading :

(a) The fundamental frequency is affected very little by the pretwist.

(b) If a chordwise frequency and a flapwise frequency are very close to eachother, the effect of pretwist is to separate them more.

(c) If the depth to width ratio is very large (say more than 20) the second and the third bending frequencies decrease with pretwist and

(d) The effect of twist on the natural frequencies is less in tapered beams than in the beams of uniform cross-section.

2.11 EFFECTS OF TEMPERATURE

The first effect of the high temperatures is a deterioration of the materials of blading. Hoff [145] has collected a great deal of information on the variation with temperature of the yield stress, the ultimate stress, and Young's modulus of elasticity.

An equally important second effect, which can often be observed at temperatures that do not impair the allowable stresses, is the development of thermal stresses. Gache and Ronzin [146] have shown that the magnitude of vibratory stresses is influenced not only by the magnitude of the quantitative magnitude of the circumferencial non-uniformity of turbine inlet temperature field but also by nature of the non-uniformity. Gorelkin and Bogov [147] considered a solution of the thermoelastic problem for gas turbine rotor blades

and presented the results of theoretical and experimental investigations of thermoelastic stresses in these blades.

A third effect of the high temperatures is that the material begins to flow slowly under comparitively small stresses ; this flow is designated as creep. In a complex structure the stress distribution is governed by the equations of equilibrium and the conditions of the compatibility of the deformations. When the latter are established with the aid of Hooke's linear law of elasticity, the ensuing stress distribution is entirely different from the one that prevails when the connection between stress and strain is given by a highly non-linear creep law.

Moreover the creep deformations are dependent on time. Depending on the material, stress, and the temperature, the time elapsing between the load application and fracture the so called critical time, may be a few years, a few days, or even a few seconds.

Turbine blades are subjected to a more severe and complex combination of stress and temperature than any other jet engine component. The blades in current engines are subjected to very high centrifugal stresses at temperatures of the order of 1200°C. Also they are subjected to hot corrosive atmosphere and to gas impulses that may cause the blades to vibrate. Rapid heating and cooling induces thermal stresses in them. Hence turbine blade failures are very common and occur in a variety of ways.

In addition to the centrifugal load, the gas forces impose a bending load on the blade airfoil. The designer partly compensates for the gas load by tilting the airfoil slightly downstream so that centrifugal force will induce opposite bending. The bending force can be completely cancelled at only one plane along the airfoil span. Further since the gas bending load reduces with altitude and engine operates at essentially constant speed, the gas bending load can be cancelled for only one altitude. If the blade is tilted to compensate for the sea level gas load, it will be overtilted for the gas load at altitude.

2.11.1 STRESS-RUPTURE

In the range of temperatures in which turbine blade operate, there is, for each blade material, a finite time before fracture for each combination of stress and temperature. In general, the life predicted from stress-rupture properties is the longest that the blade could be expected to run at full engine power, since the only stress considered is that induced by a centrifugal force. Other environmental factors, such as vibratory stress, corrosion, thermal stress and impact, will reduce the life below this value.

Incipient stress-rupture failure appears as irregular intergranular cracking in a narrow zone of airfoil span known as ''Critical zone ''. Usually cracks are not confined to the leading or trailing edge, but occur at random across the

chord. Blades that have completely fractured have many cracks on the airfoil surface adjacent to the fracture. Since the fracture is intergranular, the fracture surface will generally be rough.

Linask [148] conducted an analytical study using Fracture machanics principle to model turbine airfoil cracking. He found that crack initiation can be related to calculated residual strains in the airfoil coating and that coating properties are an important consideration in determining crack location and orientation. The coating crack subsequently propagates into base material according to basic fracture machanics laws.

Detection of stress-rupture cracks during engine inspection cannot be counted upon as method of avoiding blade fracture, because complete fracture generally follows cracking very shortly.

2.11.2 <u>CREEP</u>

At any condition of stress and temperature on the stress-rupture curve, the test specimen elongates with time in a manner described by the ideal curve of Fig.(2.11.2). When load is applied, the specimen alongates elastically then plastically at a decreasing rate (first-stage creep) until the rate becomes approximately constant (second stage). Finally the rate begins to increase (third stage) until fracture occurs.

Unit elongation or strain, is not uniform along a blade length, because of the nonuniform stress and temperature conditiond. The zone of maximum creep corresponds to zone of minimum stress - rupture life.

The designer is interested in knowing the allowable centrifugal stress and temperature because they affect both the time to rupture and the time in which elongation of the blade will deplete the operating clearance and cause the blade tip to rub shroud band. Because the strain in a blade is highly localized, total elongation is usually small ; stress-rupture fracture usually occurs before the allowable total elongation is exceeded. Thus, stress-rupture usually becomes the more important design criterion.

Creep appears as blade stretch, blade- shroud rubbing, or blade necking (local reduction in area). Necking cannot be relied upon to indicate incipient blade fracture because the time between its first appearance and fracture is usually too short and unpredictable.

2.11.3 THERMAL STRESS FATIGUE

Large temperature gradients occur in turbine nozzle vanes and blades during transient conditions such as start up , acceleration, and shutdown. Since the leading and trailing edges of turbine blades heat and cool much more rapidly than the thick midchord region, chordwise temperature differences occur in the blades whenever the engine is operated

through transients. For example, in a normal start on one engine, a difference of 500°C was measured between the leading edge and the centre of a turbine blade about 8 seconds after ignition.

Temperature differences induce stresses in the blades because of differential thermal expansion. The heavier body section of the blade restrains the expansion of edges, and thus high compression stresses (during heating) or tensile stresses (during cooling) are induced in the edges. With the result that plastic flow may occur in these areas. When the body eventually attains the equilibrium temperature, the direction of the thermal stress in edges which have been plastically deformed is reversed. This phenomenon is repeated during successive cycles of engine operation and has caused warping and cracking in edges of nozzle vanes and blades.

2.12 FATIGUE

Periodic forces resulting principally from disturbances in the gas flow through the engine cause vibration of such items as turbine blades, compressor vanes and blades. When the frequency of the vibratory force is in resonance with a natural frequency of a component and the damping losses are small, high stresses can result which may cause early failure of the component.

2.12.1 COMPRESSOR BLADES FATIGUE

One of the most vulnerable components in a gas-turbine

unit is the axial-flow compressor, Vibration fatigue account for many compressor losses.

Fatigue usually occurs in the rotor blades, but is occasionally experienced and the stator blades or rotor disks at the blade recess because of vibratory loads from the blades. The blade vibrations are generally induced by pulsations in the airflow. In most cases of blade failure, the fractures occur in the airfoil section near the base and show progressive damage typical of pure fatigue. The origin of the failure is usually on the convex surface at the maximum camber point, where the bending stress is maximum during vibration. Fatigue failures may also originate in the trailing edge of the airfoil. In other cases, the failure is in the fastening. Frequently fatigue failures originates at random locations on the airfoil where stresses are not ordinarily high but are magnified by stress concentrations arising from toolmarks or nicks made by foreign objects. Six types of fastenings have been used for compressors. In the fir - tree types, high stress concentrations occur in the fillet between serrations, particularly if the radii are small. The bending stresses due to vibration are superimposed on the steady centrifugal and gas-bending stresses. All stresses are magnified by the stress concentration caused by the abrupt changes in cross-section.

Fatigue results from alternating stresses produced by blade vibrations. Listed in order of importance, factors that

may induce these vibrations are :

(1) Rotating stall, (2) aeroelastic coupling.

2.12.1.1 ROTATING STALL

Rotating stall is one of the principal causes of severe vibrations leading to fatigue failures . This phenomenon occurs mainly during acceleration of the engine through the speed range from 50 to 70 percent maximum speed. The jet engine is seldom subjected to this speed range for long periods of operation, as the range is traversed only upon acceleration and deceleration. Neverthless, rotating stall constitutes the primary source of faitgue failure. Reference [149] gives means of alleviating rotating stall frequencies. These include (1) increasing the damping of the blades,(2) preventing the operation of any compressor blade row in a stalled attitude, and (3) disrupting the periodicity of the stall patterns so that a resonant condition between the blade frequency and stall frequency cannot exist.

2.12.1.2 AEROELASTIC COUPLING

The interaction of aerodynamic forces and the elastic behaviour of the blading can result in an instability which produces blade vibrations. Flutter is an instability-- self maintained by the continual absorption of energy from the airstream. In compressors, the problem may be aggravated by high oscillatory aerodynamic loading, proximity to stall, pressure rise across the stage, and certain cascading effects.

Excitations occuring in the vicinity of stall is known as Stall flutter, which is entirely different from rotating stall. Rotating stall is a periodic nonuniformity in the air-flow that only excites blade vibration when the impulse frequency approaches the blade natural frequency [150]. Stall flutter is self - induced vibrations of the blade in a uniform flow field due to aerodynamic instabilities arising from high angles of attack.

Stalling flutter has been shown to occur on compressor cascades in bench tests [151] . Experimental investigations [152] carried out on compressors under operating conditions have shown that in some cases the vibratory stresses increase greatly with angle of attack. An analysis was made of the effect of centrifugal forces on the flutter of a uniform cantilever beam at subsonic speeds with application to compressor and turbine blades by Mendelson [153]. This is dealt in more detail in section [2.13].

2.12.2 <u>TURBINE BLADES FATIGUE</u>

The most important source of excitation of turbine blade vibrations is the impulse given to the blades when they pass through the wakes of the nozzle vanes. The flow from individual combustion chambers likewise introduces irregularities into the gas flow impinging on the blades and may also excite vibrations. The frequency of vibratory impulse imparted to the blade is thus a function of the number of nozzle partitions, or cumbustion chambers, and of

engine speed. The blade can vibrate in simple bending, in torsion, or in complex combination.

When the vibrating stress is superimposed on the centrifugal stress the appearance of the fractures is dependent upon the magnitude of the vibratory stress relative to mean tensile stress. This is shown in reference [154].

A combined creep and fatigue problem may arise in aircraft gas turbines, where the turbine blades are subjected to centrifugal stresses together with fairly high frequency (1000 Hz) bending fatigue stresses caused by the hot gases impinging on the blades ; temperature may be above 1000°C . The rotor disks are subjected to repitetion of tensile stress for several hours, together with peak stresses for short times. High local stress concentrations are set up at cooling holes and fir - tree blade root fixings, and the material must be capable of with standing these for approximately 10000 flight cycles. It is stating the obvious when it is noted that the failure of blades or disks may have disastrous effects on the safety of the aircraft. Ellison [155] has discussed modes of failures and various methods of testing where there is interaction between creep and fatigue.

Majumdar [156] has analysed all the major components of the gas turbine engine for low cycle fatigue and /or creep. A linear cumulative damage law is assumed, since the interaction between creep and fatigue is still an unsolved problem.

2.13 AERODYNAMIC EXCITATIONS

The discussion so far was confined to the methods for the calculation of natural frequencies and mode shapes, however, from strength consideration point of view, knowledge of the amplitude of vibration is also essential. The aerodynamic exciting forces, the methods of calculating deflection amplitude, stress levels, etc. are considered.

The aerodynamics of compressors, especially axial flow, being complex, the analytical treatment of this field is somewhat limited. The practical aspects of design and vibration analysis have been reviewed in several papers. The more important of these reviews are those by Blackwell [157], Pearson [158], Armstrong and stevenson [159], Ref.[154] and Ref. [149] etc.

2.13.1 FLUTTER

The approach to flutter prediction of fans and compressors in gas turbine engines is different from the approach to flutter prediction of aircraft wings and other external aerodynamic systems. These differences arise from the greater disparity between the density of the air and the density of the structure, and from the more complex boundary to the air formed by adjacent blades and walls in the gas turbine.

In the past [Ref. 149,154,157,158,159], the difficulties of predicting the proper unsteady aerodynamic forces for cascades of blades found in turbomechines have necessitated the use of empirical approaches to flutter prediction. These

approaches tried to distinguish many different types of flutter and to correlate parameters measuring the onset of flutter with other structural or aerodynamic parameters [160-163]. Thus the onset of flutter was usually characterized by the relative velocity at some specified spanwise location on the blade, divided by the product of vibrational frequency and chord of the blade at blade at that spanwise position. This was correlated against a number of parameters like the ratio of torsional to translational deflections, the relative Mach number, or the nondimensionalized incidence angle. The parameteric plots served to correlate experimental data for different types of flutter [164-166], the onset of flutter being defined by an empirical boundary.

The empirical correlations have become inadequate as recent advances in technology (higher tip speeds in turbines, avoidance of heavy shrouded construction, etc.) required extrapolations of past experience to new areas. The sharp dividing lines between stable operation and flutter disappeared. Attempts to improve the correlating parameters were unsuccessful, due to scarcity of new data and the necessity to extrapolate outside past experience. A need to develop a unified flutter prediction based on basic aerodynamics and vibration analysis, and capable of application to all new design, became very real.

A uniform approach to flutter prediction has been developed by Mikolajczak et al. [167]. The aeromechanical stability of the blade-disk system is expressed in terms of a

stability parameter which measures the amount of unsteady work done by the air on the systam, when the system is vibrating in one of its natural modes. In neutrally stable systems, the unsteady work done by the air on the blades will balance the work dissipated by friction and by material damping. An accurate prediction of the vibrational deflections and of the unsteady aerodynamic forces is required at every spanwise location on each blade, so that the work done by the unsteady aerodynamic forces may be calculated. Recent progresss is described in the prediction of unsteady aerodynamic forces and the determination of mode shapes. The stability model is applied to the prediction of supersonic flutter, chordwise bending flutter, and stall flutter.

Flutter accurs when the unsteady aerodynamic forces and moments created by periodic blade vibrations do positive aerodynamic work on the blade during each vibration cycle and the mechanical damping is insufficient to dissipate this work input.

If the rotor system (blade, disk and shroud) is assumed to be a simple linear system, then the aerodynamic damping for system [167] can be written as

$$\delta_{aero} = - nW / 4\,\overline{KE}$$

where n is the number of blades in the rotor, and $\overline{KE}$ is the average vibrational kinetic energy of the rotor system.

Flutter is obtained when the total damping for any

vibrational mode is less than zero, i.e.,

$$\delta_{total} = \delta_{aero} + \delta_{mech} < 0$$

The mode with the lowest damping is assumed to be the least stable. For flutter prediction, it is usually sufficient to examine only the lower frequency modes, namely the first three modes, since higher modes tend to have a high positive damping. Of course, it is necessary to examine the higher modes, so called plate modes, using chordwise flutter predictions.

Boundaries for the most common types of flutter are shown schematically on a compressor map in Fig. [2.13.1]. The distinguishing features of each type of flutter prediction are summarized as follows [168].

(1) <u>Supersonic unstalled flutter</u> : This type of flutter occurs when the outer span of the blade is operating supersonically and can occur at the design point or at either higher or lower pressure ratios. Its occurence imposes a limit on high speed operation. This type of flutter usually occurs in vibration modes which are predominantly torsional [165].

(2) <u>Subsonic stalled flutter</u> : This type of compressor flutter is encountered near the surge line. In a high-speed fan stage, subsonic stalled flutter may occur at part speed ; in a low - or high - pressure compressor, subsonic stalled flutter may occur at or near design speed. It is suspected that during stalled flutter, separated flow may exist on the compressor blades.

(3) Choke (negative incidence)flutter : This type of compressor flutter may occur when a fan is operating either subsonically or transonically. The stresses will increase as pressure ratio is lowered. During choke flutter, the rotor is operating at near - choke conditions and the flow is transonic over most of the blade chord.

(4) Supersonic stalled flutter : This type of compressor flutter occurs when the outer portion of the blade is operating supersonically, and the stage is operating near the surge line. The flow has strong shocks in the blade passages.

Stalled flutter and supersonic unstalled flutter are the two most common types of flutter.

Chordwise Blade Flutter

For very high speed fans for which composite materials are currently being developed, supersonic unstalled flutter can occur in a vibratory mode which involves a large vibrational deformation of the blade chamber line. Such modes are present and important near blade tips when the rotor aspect ratio is low and the blades are thin. It can be seen that the susceptibility to flutter increases with Mach number.

Stalled Flutter

Stalled flutter usually occurs near the compressor surge line and is identified by an increase in flutter stress as pressure ratio is increased at constant speed.

In some instances the stall flutter boundary has been

observed to occur near maximum compressor efficiency indicating that stalling is not an essential condition for ''stalled'' flutter, although it could be the most severe condition. This indicates that the influence of flow incidence on the unsteady aerodynamic forces will have to be considered in the prediction of the flutter boundary.

2.13.2 WAKE EXCITATION

Rotating blades provides some of the most challenging problems of fluid mechanics. Flow within a turbomachine is complex, three dimensional, rotational, asymmetric, compressible viscous and unsteady.

Wake excitation phenomenon gives rise to an unsteady pressure distribution and lift force on each blade is a source of mechanical vibration as well as acoustic radiation. The problem is essentially one of unsteady airfoil theory, i.e., an airfoil travelling through a sinusoidal gust (velocity perturbation). Naumann and Yeh [169] develops the unsteady pressure and lift for an airfoil that has both camber and angle of attack, under longitudinal as well as transverse velocity perturbations, ignoring the interference effects due to adjoining blades and hence is valid for cascade of low solidity.

Physical mechanisms (e.g. self - excited vibrations of rotor blades and stator vanes, shaft and disk vibrations etc.) are discussed [170] as possible aeromechanical responses to

distorted flow. The role of various forms of damping and the use of composite materials is also described.

Denforth [171] has described 8 mechanisms of distortion induced vibrations. A distortion index for blade vibrations is defined both as an initial vibration design alert and for automatically assuring integration of design for inlet aerodynamics with that for blading. These 8 mechanisms are : (1) stall, (2) resonance , (3) random vibration generated by excitations associated with cascade separation, (4) its combination with resonance maxmized at a critical operating temperature, resonance indirectly generated by distortion accentuation of these mismatched stator cascades, (5) blades instability, or flutter, induced by distortion in stages stable without distortion, (6) inlet gust excitation, (7) upstream turbulence, and (8) oscillatory inlet flows. Discussion is limited to blade vibration in the prestall operating regime; of fans and compressors ; for stall, referred to fans and compressors as a whole rather than local ''cascade stall'', is a generally unacceptable operating conditions. Blade response in fan or compressor stall is a specialised subject in itself.

2.14 ASPECT RATIO

Much of the early work carried out in studying the vibration characteristics of turbine blading was based on the assumptions of large aspect ratio. Carnegie [36], Esakson

and Eisley [29], and Houbolt and Brooks [47], among others, used simple beam theory and various solution techniques to study the vibratory behaviour of twisted cantilever beams as an analog for twisted propellors and turbine blading. While these works are complete in their own right, their techniques can not adequately handle the newer lightweight, low aspect ratio turbine blading where the blades are more likely to behave as plates or shells rather than beams. The solution to this dilemma arrived with the advent of finite element analysis methods. Now, a blade having a complex geometry can be accurately modeled as a composition of many elastic elements having well-defined elastic properties. Anderson, Irons, and Zienkiewicz [172] aptly demonstrated the utility of finite element analysis in studying the vibration and stability behaviour of thin plates for various geometries and boundary conditions. Rawtani and Dokainish [173,174] have very effectively used finite element techniques to study the vibration behaviour of twisted cantilever plates. They analyzed the effect of varying degrees of pretwist on the natural modes and frequencies of cantilever plates [173]. They also carried out a vibration analysis of a flat cantilever plate fixed to a rotating disk [174]. The effects of plate inclination, disk radius, and angular velocity on the plates natural frequencies and mode shapes were studied.

Dokainish [175] has extended the transfer matrix technique generally known as the ''Holzer-Myklestad'' method,

to reduce the size of matrices in the finite element method used in the vibration analysis of plates and shells. As such the transfer matrix has been applied earlier by Leckie [176] to plate vibration problem.

Macbein [177] have presented a combined numerical - experimental study of the effects of varying tip twist and increasing centrifugal loading on the resonant characteristics of cantilevered plates. The finite element computer program NASTRAN, is used to compute the natural frequencies, mode shapes, and normalized shear stress distribution.

2.15 EFFECTS OF SHROUD

Turbomachinery blades are often joined together, either at their tips or at some intermediate location, to reduce the resonant amplitude. A single shroud ring may join all the blades in a row or else a packet of blades may be joined together. The continuous shroud is broken between groups of blades with a gap of order of 0.5 mm to allow for its expansion. The shrouding significantly alters the natural frequencies and mode shapes and hence the packet of shrouded blades should be analysed as a unit for the tangential, axial or torsional vibrations. The shroud also causes a coupling between the axial flexural vibrations and the torsional vibrations.

The effect of shrouding on the tangential vibrations (vibrations in the place of the disk) of a packet of blades

is investigated by Smith [178] using several simplifying assumptions. Prohl [179] considered not only the tangential vibrations but also the axial and the torsional vibrations. A method of finding the natural frequencies of a group of shrouded blades, using the perturbation procedure is given by Tuncel et al [180]. The method is applicable for weakly coupled blade and shroud subsystems. Fugino [181] has considered the effect of shrouding by taking the vibration form of the blades as a linear combination of fixed-free, fixed-supported and fixed-fixed modes for the tangential vibrations.

The problem of two adjacent blades, connected at their tips by a shroud to form a continuous frame, has been analysed by Singh and Nandeoswaraiya [182] . The fundamental frequency is obtained for the tangential and the axial vibrations. For rotating blades, a Southwell type correction factor is also derived.

An investigation of the vibrations of a coupled disk-blade-shroud assembly is considered by Stargardter [183], by experiments on flexible silicon rubber models. It observes that the disk vibrates as a plate, mainly with modal diameters, and the shroud acts as ring. Bhide [184] and Bajaj [185] have also studied the vibrations of packetted turbine blades.

Shroud design is effective against untwist [186]. Not

only turbine blades, but almost all of the large fan blades have part-span shrouds to get better vibration characteristics and these shrouds stand against untwist deformation by constraining themselves eachother.

2.16 LACING WIRES

One of the reasons for using shroud is to hold blades in the correct position. This is also achieved by Lacing or Lashing Wires. Shrouding introduces high centrifugal forces where as Lacing achieves the purpose of alignment without appreciable addition to centrifugal forces.

Formerly lashing wires were strung through the bossed holes, machined in the blades. The present practice is to weld short pieces of lashing wires to each side of the blade. When the blades are machine-finished and stresses are relieved, these lashing wires touch each other and are joined either by a weld or by a welded sleeve. The main disadvantage of lashing wires is that it disturbs the flow pattern and contributes to vibration tendencies. To avoid these, airfoil section is used for the lashing wires. Lashing wires could be provided at two or three places along the length of the blade in order to give it sufficient rigidity.

2.17 LARGE AMPLITUDE

All most all the papers discussed earlier assumes that amplitudes are small. In dealing with large amplitude vibration of deformable bodies such as beams, plates and shells

the relationship between extensional and shear strains on the one hand and the displacement components on the other are assumed to be non-linear. Thus we have geometric non-linearity and as a consequence the governing differential equation will turn out to be non-linear. This is true inspite of the fact that the relations between curvatures and displacement components are assumed to be linear. (This is also called as Structural non-linearity -- due to large amplitude causing stiffening of the structure thus raising the frequency of response).

This need not be the case if the non-linearity is due to material (e.g. Ramberg- Osgood material).

Another way of looking at this is to consider the total strain energy of the deformable elastic body which is the sum of the extensional or stretching energy and bending energy. The extensional energy becomes non-linear (involving higher order non-linear terms than quadratic) in the normal displacement component, whereas the bending energy remains quadratic in the displacement components.

Srinivasan [187] has considered non-linear vibration of beams and plates. Saythyamorthy and Pandalai [188] has obtained governing equations for large amplitude flexural vibrations of plates and shells. Large amplitude vibration of deformable bodies are considered in Ref. [189].

2.18 DAMPING

The damping in a system determines the amplitude, and

consequently the vibratory stress, for a given exciting force. A material may have very high fatigue strength, but if the damping is very low, the material may have a much shorter life than one with moderate fatigue strength and high damping for the same exciting force. Material damping is the most reliable of all the sources available. Root damping may vanish because of high centrifugal force or malfunctions of an external damper. Aerodynamic damping may vanish because of unfavourable coupling between the aerodynamic and elastic forces. Some materials display increased internal damping as the vibratory stress increases Ref. [154].

Various types of damping can be employed to minimize the resonant amplitudes in order to reduce roughness, noiseness and susceptibility to fatigue failure. The principal sources of dissipation of energy from the blade are :

(a) Friction at the root and the hub ,

(b) Internal damping due to inelasticity of the material, and

(c) Damping due to the presence of surrounding gas (Aerodynamic Damping).

The dissipation of energy at the root due to the relative motion between the blade and the rotor in mechanical attachments. At lower rotor speeds this energy dissipation may be appreciable, however, at very high rotor speeds the centrifugal force essentially tightens the blade

sufficiently to eliminate the dissipation. Some analysis of this type of damping has been reported in Refs. [190-192]. Goodman and Klump [191] have pointed out that when blade root and hub are joined in press fit, the energy is dissipated by the microscopic slip at the interfaces. An experimental determination of the damping at the locking joints, when the turbine blades are vibrating, has been carried out by Kozolov [192]. By investigating the different types of locking joints, the important parameters affecting damping have been determined.

Although these investigations offer some understanding of the phenomenon of root damping, it is very much dependent on the type of the root, manufacturing tolerances, speed of rotation etc. Hence it is difficult to account for it in the analytical calculations.

The material damping must often be relied upon, where structural slip and other types of external damping are not effective [193]. The material damping is caused by inelasticity of the material. The stress strain curve for loading and unloading are not generally identical but form a hysterisis loop. The area bounded by the loop is a measure of the energy dissipated in each cycle. A brief resume of the damping capacity of the materials is given by Lazan [194]. Some investigation of the effect of internal damping in rotating beams has been carried out by Morduchow [195].

An extensive study of the flexural and torsional vibrations of beams, with a special reference to turbine blades, taking into account the internal damping has been carried out by Pisarenko [196]. Baker et al. [197] have considered the vibrations of cantilever beams in surrounding air. The damping arising from the air drag as well as from the internal material hysteresis have been taken into account.

Effect of internal damping on the transverse vibration of non-uniform cantilever beam is considered by Kerlin [198]. Glnin and Maybee [199] has analyzed the effect of internal material damping on the stability of rotating systems.

The aerodynamic damping is caused by the fact that the velocity acquired by a vibrating blade changes the incidence of the air stream. If the cascade characteristics are such that an increase of incidence causes an increase in the lift force on the blade, damping takes place. However, if an increase of incidence causes a decrease in the lift, it results in flutter, Pearson [200] has pointed out that in the unstalled region, the damping present is almost wholly aerodynamic, even when the excitation itself is almost wholly aerodynamic, e.g. from wakes, etc. He has derived an expression for the energy removed by the airstream from a vibrating blade per unit time. Cavaille et al.[201] have presented a study of aerodynamic damping of turbomachine

blade vibration.

Jones et al. [202] have described vibrating beam dampers for reducing vibrations in gas turbine blades. Parin and Jones [203] present a preliminary investigation of a tuned damper utilizing a polymeric damping material encapsulation to withstand centrifugal load and erosion problems.

2.19 MATERIAL

The gas turbine imposes on materials the most severe condition known to date. Beside extreme mechanical strength at temperatures upto approx. $1000^{\circ}C$, corrosion resistance to a highly reactive environment must also be maintained. This controversial combination of properties has posed a challange that has been successfully met by material scientist, has been done since World War II, particularly on Jet aircraft turbine.

Improvement in the physical properties of materials have always been enthusiastically received by the aeronautical engineer. Basic knowledge in aerodynamics and structures has always been ahead of the metallurgical developments, and the designer can always make use of improved materials as soon as they become available. This situation is particular true at the present time. The lack of materials having the required physical properties at elevated temperatures seems to be the most important handicap in the development of supersonic

aircraft gas turbine, and progress in this direction could be seriously hampered if research in the field of high temperature materials are not very actively pursued.

The field of high temperature materials is by no means restricted to those materials having very high melting points. High temperature problems in materials exist over entire range of temperature.

The report [204] gives the current state - of - the art of material usage in aircraft gas turbine engines, including auxiallary power units, and discusses the trends in future materials such as composite, powder metullargy, controlled solidification, eutectic alloys, refractory metals and ceramics. New processing techniques, such as thermo-mechanical processing (TMP) and Gatorizing TM are also discussed.

The tendencies manifesting themselves in the evalution of alloys which are currently used in the construction of turbines are reviewed [205]. The techniques of powder metullargy and oriented solidification should make it possible to make better use of Ni alloys. The perspectives of application of composite materials are examined. If the resistance to impact of these materials appears insufficient in case of moving compressor blades, their other mechanical characteristics permit a major lightening of other parts less exposed to shocks. Oriented eutectics, alloys of particular composition which take up a composite material structure

I.I.T. KANPUR
CENTRAL LIBRARY
Acc. No. A 47186

by unidirectional solidification present a set of properties which leads to expectation that some will be used for making turbine blades or vanes. Huff and Betz [206] also considered directionally solidified eutectic alloys and their application to turbine blades and gas turbine components. They conclude that lot of further investigation is necessary especially with respect to casting technology.

Jahnke and Brnch [207] also contend that composite structures consisting of high strength fibres or plates in ductile materices with outstanding high temperature properties are achievable in directionally solidified eutectics. This new class of materials represent a major innovation in gas turbine blade technology. The advantage and limitations of two more promising eutectic system and the relationship of these properties to turbine blade design are discussed.

Endres [208] has taken design examples of some typical high temperature parts such as vanes, blades and segments shielding hot surfaces and showed what conditions are encountered during service.

The results of impact test [209] on large, solid fiber composite (graphite and boron/epoxy) fan blades for aircraft turbofan engine are discussed. It also showed that composite blades may be able to tolerate iceball and small bird impact. Nabotova and Shipov[210] carried out a series of tests with geometrically similar steel and

duralumin rotor blades on a compressor stage model. It is shown that, with the same level of blade frequency differences, the rotor with steel blades has approximately 2.8 times higher critical pressure at which flutter arises than the rotor with duralumin blades.

A Finite element method of rotating ceramic gas turbine blade consisting of an airfoil and root is discussed [211]. Three dimensional results using both isotropic and orthotropic material properties and two dimensional study of root considering the effect of friction, contact area and root geometry is included.

Schallar, Visser and Lien [212] has presented a three-dimensional analysis of a ceramic rotating blade. The analysis reveals that under a centrifugal force load the maximum stress occurs in the root of the blade. The root, a critical area for this blade, is then analyzed in detail in an effort to reduce this stress. This paper is the first step in a feasibility study to ascertain if ceramics are applicable to rotating blades for heavy duty gas turbines.

Gaytor [213] has presented wide range of creep resistant nickel - base alloys now available and developed primarily for use at elevated temperatures in gas turbine engine.

CHAPTER III

FORMULATION OF THE PROBLEM

3.1 Introduction

In this chapter the transfer matrix approach is used to determine the mode shapes and natural frequencies of a turbine blade. The low temperature side of the turbine can be idealized as a large aspect ratio cantilever beam tapered in both depth and width. Some of the predominant factors like rotation, shear deflection, rotary inertia, root flexibility, stagger angle etc., which affect the response of the blades are considered. 'State matrix' is derived for a small element of the beam and this is used in the determination of the elements of transfer matrix. The Runge-Kutta method is used for finding the transfer matrix for the beam subjected to pure bending.

3.2 Formulation of State Matrix for Pure Bending Mode

The slope $\frac{dw}{dx}$ of the centre line of the beam is affected by the bending moment M and the shear force V. The angle between the perpendicular to the face and the centre line of the beam is caused by the shear force acting on the beam. Inspection of Fig. (3.2.1b) shows this angle to be equal to $(\frac{dw}{dx} + \psi)$. The relation between this angle and the shear force causing it is

$$V(x) = GA_s(x) \left(\frac{dw}{dx} + \psi \right) \qquad (3.2.1)$$

where $GA_s(x) = GA(x)/K_s$ is the shear stiffness, K_s being a shape factor depending on the shape of the cross-section, $A(x)$ is the area of cross-section.

The usual bending relation for a beam is

$$M(x) = EI(x)\,\frac{d\psi}{dx} \qquad (3.2.2)$$

Now the rotating beam is subjected to distributed centrifugal force field. Considering an element dx dy dz with coordinates (x,y,z) subjected to components of the centrifugal force per unit volume along x,y and z axis are given by Ref.[174].

$$F_x = \rho\,\Omega^2\,(x+r)$$
$$F_y = \rho\,\Omega^2\,(y\,\mathrm{Cos}^2\theta - z\,\mathrm{Sin}\theta\,\mathrm{Cos}\theta)$$
$$F_z = \rho\,\Omega^2\,(-y\,\mathrm{Sin}\theta\,\mathrm{Cos}\theta + z\,\mathrm{Sin}^2\theta)$$

where ρ is the mass per unit volume of the beam.

Now these above components are integrated between limits $y = -D/2$ to $D/2$ and $z = w-\frac{W}{2}$ to $w+\frac{W}{2}$ to obtain the forces on the beam element of length dx situated at x from root. These are

$$dF_x = \int_{-D/2}^{D/2}\int_{w-\frac{W}{2}}^{w+\frac{W}{2}} F_x\,dx\,dy\,dz$$

$$dF_y = \int_{-D/2}^{D/2}\int_{w-\frac{W}{2}}^{w+\frac{W}{2}} F_y\,dx\,dy\,dz$$

$$dF_z = \int_{-D/2}^{D/2}\int_{w-\frac{W}{2}}^{w+\frac{W}{2}} F_z\,dx\,dy\,dz$$

or

$$dF_x = \mu(x)\,\Omega^2\,(x+r)\,dx$$
$$dF_y = -\mu(x)\,\Omega^2\,w\,\mathrm{Sin}\theta\,\mathrm{Cos}\theta\,dx$$
$$dF_z = \mu(x)\,\Omega^2\,w\,\mathrm{Sin}^2\theta\,dx \qquad (3.2.3)$$

where $\mu(x)$ is mass per unit length.

It is assumed that taper in both depth and width is linear,

$$D(x) = D_o - (D_o - D_{NS})\frac{x}{L} \text{ ; Depth variation}$$

$$W(x) = W_o - (W_o - W_{NS})\frac{x}{L} \text{ ; Width variation}$$

where D_o, D_{NS}; W_o, W_{NS} are depths and widths of the beam at the fixed and free ends respectively.

Force balance gives (Ref.fig.3.2.2) -

$$-R+(R+dR)+dF_x = 0$$

$$dR = - dF_x$$

and

$$V-(V+dV)-dF_z.\cos\psi - \mu(x)\omega^2 w(x).dx = 0$$

For small values of ψ,

$$\frac{dV(x)}{dx} = -\mu(x)\omega^2 w(x) - \mu(x)\Omega^2 w(x) \sin^2\theta$$

or

$$\frac{dV(x)}{dx} = -\mu(x) (\omega^2 + \Omega^2 \sin^2\theta).w(x) \quad (3.2.4)$$

Moment balance gives

$$M-(M+dM) - (R + dR).dw - dF_x.\frac{dw}{2} + (V+dV).dx + [dF_z+\mu(x)\omega^2 w(x).dx].\frac{dx}{2} - \mu(x)i_y^2(x)\omega^2\psi(x).dx = 0$$

Neglecting second-order terms and arranging,

$$\frac{dM(x)}{dx} = V(x) - R(x).\frac{dw}{dx} - \mu(x)i_y^2(x)\omega^2.\psi(x) \quad (3.2.5)$$

where

$$R(x) = \int_x^L \mu(x)\Omega^2 (x+r)\, dx \quad ,$$

the tension in the beam element due to rotation.

Rewriting these equations as

$$\frac{dw(x)}{dx} = -1 .\ \psi(x) + 1/GA_s(x) .\ V(x)$$

$$\frac{d\psi(x)}{dx} = 1/EI(x) .\ M(x)$$

$$\frac{dM(x)}{dx} = (1-R(x)/GA_s(x)) .\ V(x) + [\ R(x) - \mu(x) i_y^2(x) \omega^2\ ] .\ \psi(x)$$

$$\frac{dV(x)}{dx} = -\mu(x) \ (\omega^2 + \Omega^2 \sin^2\theta) .\ w(x) \qquad (3.2.6)$$

where μ is mass per unit length, ω is frequency of vibration of rotating beam, Ω is speed of rotation, r is radius of the disk, L is length of the beam and i_y^2 is radius of gyration.

When the state equations (3.2.6) are written in the matrix form :

$$\frac{d}{dx}\begin{bmatrix} -w(x) \\ \psi(x) \\ M(x) \\ V(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & Q(x) \\ 0 & 0 & P(x) & 0 \\ 0 & RN(x) & 0 & RM(x) \\ AM(x) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -w(x) \\ \psi(x) \\ M(x) \\ V(x) \end{bmatrix} \qquad (3.2.7)$$

where

$$RM(x) = [\ 1 + R(x) .\ Q(x)\ ]$$

$$
\begin{aligned}
AM(x) &= \mu(x)\ (\omega^2 + \Omega^2 \sin^2\theta) \\
N(x) &= -\mu(x)\ i_y^2(x)\ \omega^2 + R(x) \\
P(x) &= 1/EI(x) \\
Q(x) &= -1/GA_s(x)
\end{aligned}
$$

or

$$\frac{d}{dx}\ [z(x)] = [\ B(x)\ ]\ [z(x)] \qquad (3.2.8)$$

where [z(x)] is the state vector and [B(x)] is the matrix of variable coefficients of system of first-order differential equations (3.2.6).

The above state vector and the state matrix would be used in the formulation of transfer matrix method which follows in the next section.

3.3 TRANSFER MATRIX

A. Description

The matrix which connects the properties (the elements of the state vector) at 'n' th section to that of the properties at 'n + 1' th is called a transfer matrix. The name 'transfer matrix' itself implies that to those of the properties of one section are related to another section through that matrix. This manner of description is eminently suited to describe the performance of a series of structures connected in tandem ; that is, output of the first is the input to second, and so on. The transfer characteristics of such a series is found by multiplication of the transfer matrices of the individual

structures, i.e., local transfer matrices.

Transfer matrix applies only to the case of one - dimensional structures, i.e., structures depending on just one spatial co-ordinate, and of certain two-dimensional structures where separation of spatial variables is possible. Consider a beam undergoing a harmonic motion, the state of motion at a given station on the beam may be described by a vector.

$$Z\, e^{i\omega t} = \begin{Bmatrix} w \\ \psi \\ M \\ V \end{Bmatrix} e^{i\omega t}$$

If excitation is about from the interval between stations a and b on the beam, then

$$Z_b = T_{b \leftarrow a}\, Z_a \qquad (3.3.1)$$

$T_{b \leftarrow a}$ is transfer matrix associated with the structural element connecting station a to station b, a 4 X 4 matrix in this case. For the transfer of k dimensional state vectors (k = even, for structural engineering), the transfer matrix is k X k.

When excitations are introduced at a station, say at p, the value of the state vector becomes discontinuous at p. If p lies between a and b, equation (3.3.1) must be modified to account for the discontinuity. Thus,

$$Z_N = T_{N \leftarrow o}\, Z_o + T_{N \leftarrow p}\, Y_p \qquad (3.3.2)$$

where Y_p is the input at p, o and N are end stations.

Since two components of Z_o and two of Z_N are zeros (upon imposing boundary conditions, this is valid even when an end is, e.g., elastically constrained by a spring, in which case the end station is considered to be immediately beyond the elastic spring, i.e, the transfer matrix, T must include the effect of elastic spring), equation (3.3.2) can be used to determine uniquely Z_o and Z_N. Knowing these we can return to equation (3.3.1) and compute the response at any station on the beam.

The impedance, admittance, receptance and transfer matrices contain identical information, but in different arrangements and their interrelationships are given in Ref. [15].

B. Some Important Properties of Transfer Matrix

(a) Cross - Symmetry

If the structural element represented by a transfer matrix is symmetrical about centre of element, by a suitable choice of the order and sign convention for the components of a state vector, it is always possible to obtain a transfer matrix symmetrical about its cross-diagonal (that is, the diagonal connecting $t_{k,1}$, $t_{k-1,2}$ ...., $t_{1,k}$ where t_{ij} are the entries of the transfer matrix). This cross-symmetic property can be expressed as

$$t_{i,j} = t_{k+1-j,\ k+1-i} \tag{A}$$

(b) Inversion

From Eq.(3.3.1),

$$T_{a\leftarrow b} = T^{-1}_{b\leftarrow a} \tag{B}$$

This means that the inverse of a transfer matrix is also a transfer matrix, and that the pair may be viewed as two propagation mechanisms in opposite directions provided by the same structural element. Furthermore, the existence of an inverse guarantees that a transfer matrix is non-singular. Dropping the subscripts of transfer matrices for brevity and denoting the elements of T^{-1} by $\bar{t}_{ij}$, then the inversion of a cross-symmetric transfer matrix merely amounts to change of signs for certain elements, namely

$$\bar{t}_{ij} = (-1)^{i+j}\ t_{ij} \tag{C}$$

For non-symmetrical structural elements for which transfer matrix is no longer cross-symmetric, (C) becomes

$$\bar{t}_{ij} = (-1)^{i+j}\ t_{k+1-j,k+1-i} \tag{D}$$

(c) Value of Determinate

It follows from Eq.(C) or more generally Eq.(D) that

$$|T| = |T^{-1}| = 1 \tag{E}$$

(d) Characteristic Equation and Eigenvalues

By use of Eqns. (D) and (E), it can be shown that the characteristic equation of a transfer matrix T and its

inverse T^{-1} are identical and they can be written as

$$\sigma^k - 2K_1\sigma^{k-1} + 2K_2\sigma^{k-2} + \ldots\ldots + 2K_2\sigma^2 - 2K_1\sigma + 1 = 0$$
$$\bar{\sigma}^k - 2K_1\bar{\sigma}^{k-1} + 2K_2\bar{\sigma}^{k-2} + \ldots\ldots + 2K_2\bar{\sigma}^2 - 2K_1\bar{\sigma} + 1 = 0 \quad (F)$$

Here k is even for structural engineering applications. Therefore T and T^{-1} have the same set of eigenvalues. Since for every eigenvalue σ_i of T there is a corresponding eigenvalue $\bar{\sigma}_i = 1/\sigma_i$ for T^{-1}, we further conclude that the complete set of eigenvalues of T(or of T^{-1}) must be pairwise reciprocal.

(e) <u>Delta - matrix of a Transfer Matrix</u>

The Δ matrix of a transfer matrix also possesses property (a) stated earlier, i.e., the Δ matrix of a cross-symmetric transfer matrix is cross-symmetric. However, its inversion does not enjoy the same relationship as that in Eq. (D). Instead it satisfies.

$$(T^{\Delta})^{-1} = (T^{-1})^{\Delta} \qquad (G)$$

so that $|T^{\Delta}| = |(T^{\Delta})^{-1}| = 1$, that the characteristic equation of T^{Δ} has a symmetrical form of the type of Eq.(F) and that the eigenvalues of T^{Δ} also form reciprocal pairs.

C. Frequency Equation and Mode Shapes

The transfer matrix, by definition, is independent of boundary conditions and is a function of frequency, ω, Upon substituting appropriate boundary conditions for the system into the state vectors of equation (3.3.1), a

frequency determinant can be obtained.

The systems now discussed are 'open' systems in the sense that there have been two sets of B.C's. If the system is closed, however, the boundary conditions are not so obvious. Suppose that we start at an arbitrary point of the closed system, where the state vector is z_o. With the help of transfer matrices it is possible to proceed around the system until we return to the original point from which we started. In this way a relation of the form

$$z_o = U_n U_{n-1} \cdots\cdots U_i \cdots\cdots U_2 U_1 \; z_o = T_1 z_o$$

or

$$(T_1 - I) z_o = 0$$

is obtained, so that the frequency condition is

$$| T_1 - I | = 0$$

The frequency determinant is, therefore, of order n of the system and not of order n/2 , which is the case with 'open' systems.

Having determined the natural frequencies, the coresponding normal mode shapes can also be obtained. This is accomplished by relating one of the non zero components of the state vector at free end to 1.0 to provide a check on the mode shape computations. Then the state vectors are determined at other points through the system, by applying the transfer matrix.

D. Derivation of Transfer Matrix by the Runge-Kutta Method

The Runge-Kutta method is well suited for finding the transfer matrix, but by and large it is necessary to have the services of an electronic computer. To compute the transfer matrix relating the state vectors z_o and z_{NS}, we define the interval between o and NS into a number of small intervals. Let us suppose that between sections n and n+1 the interval size is h (Fig. 3.3.1).

The Eq. (3.2.8) is rewritten as

$$\frac{d}{dx}\,[z(x)] \;=\; [B(x)]\,[z(x)] \qquad (3.3.3)$$

The Runge - Kutta method gives the result

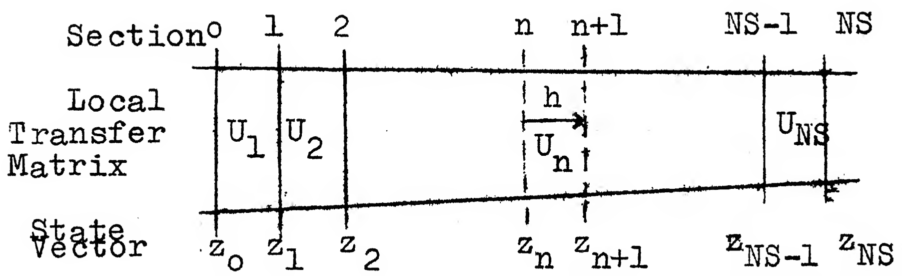


Fig.(3.3.1) The subdivision of the interval between o and NS.

$$z_{n+1} \;=\; z_n + \frac{1}{6}\,(K_o + 2K_1 + 2K_2 + K_3) \qquad (3.3.4)$$

where

$$K_o = hB(x_n)z_n \qquad\qquad K_1 = hB(x_n+\tfrac{h}{2})\,(z_n+\tfrac{K_o}{2})$$

$$K_2 = hB(x_n+\tfrac{h}{2})(z_n+\tfrac{K_1}{2}) \qquad\qquad K_3 = hB(x_n+h)\,(z_n+K_2) \qquad (3.3.5)$$

$B(x_n)$, $B(x_n+h)$, and $B(x_n+\frac{h}{2})$ are, respectively the values of the matrix $B(x)$ at sections n and n+1 and at the section midway between n and n+1.

Substituting expressions (3.3.5) into Eq. (3.3.4).

Thus the relation between z_{n+1} and z_n :

$$z_{n+1} = \Big\{ I + \frac{h}{6} [B(x_n) + 4B(x_n + \frac{h}{2}) + B(x_n+h)] + \frac{h^2}{6}[B(x_n + \frac{h}{2}).B(x_n) + B(x_n+h).B(x_n+\frac{h}{2}) + B^2(x_n+\frac{h}{2})] + \frac{h^3}{12}[B^2(x_n+\frac{h}{2}).B(x_n) + B(x_n+h).B^2(x_n+\frac{h}{2})] + \frac{h^4}{24}B(x_n+h).B^2(x_n+\frac{h}{2}).B(x_n) \Big\} . z_n \tag{3.3.6}$$

where the contents of the braces represent, the local transfer matrix U_n linking the state vector z_n and z_{n+1} . By repeating the process for all the intervals between o and NS, the overall transfer matrix T relating z_o and z_{NS} can be found by multiplication of the braces of Eq. (3.3.6) in the proper sequence. In Fig. (3.3.1) it is clearly shown that the overall transfer matrix can be obtained by successive multiplication of these local transfer matrices.

From Fig. (3.3.1),

$$z_1 = U_1 z_o \text{ , } z_2 = U_2 z_1 \text{ ,........, } z_{NS} = U_{NS} z_{NS-1} \tag{3.3.7}$$

By backward substitution z_{NS} is related to z_o.

Therefore, $$|z|_{NS} = |U|_{NS} |U|_{NS-1} \cdots\cdots |U|_2 |U|_1 |z|_o$$

$$|z|_{NS} = |T| \; |z|_o \tag{3.3.8}$$

where, $$|T| = |U|_{NS} |U|_{NS-1} \cdots\cdots |U|_2 |U|_1$$

3.4 Determination of Natural Frequencies and Mode Shapes

Writing Eq. (3.3.1) in expanded form,

$$\begin{bmatrix} -w \\ \psi \\ M \\ V \end{bmatrix}_{NS} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \begin{bmatrix} -w \\ \psi \\ M \\ V \end{bmatrix}_{o} \tag{3.4.1}$$

where the index 'NS' corresponds to the free end and 'o' corresponds to the fixed end of the beam.

Imposing the boundary conditions that is shear and bending moment would vanish at free end (NS), and deflection and bending slope would vanish at the fixed end ('o').

$$\begin{bmatrix} -w \\ \psi \\ 0 \\ 0 \end{bmatrix}_{NS} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ M \\ V \end{bmatrix}_{o} \tag{3.4.2}$$

The above equation leads to the following condition :

$$\begin{aligned} T_{33}M_o + T_{34}V_o &= 0 \\ T_{43}M_o + T_{44}V_o &= 0 \end{aligned} \tag{3.4.3}$$

Non-trivial solution of the above equations exist only when the corresponding determinant vanishes.

Therefore,

$$\begin{vmatrix} T_{33} & T_{34} \\ T_{43} & T_{44} \end{vmatrix} = 0 \qquad (3.4.4)$$

The values of 'ω' for which Eq.(3.4.4), called frequency equation, is satisfied are the natural frequencies of the system. These natural frequencies are determined by a linear interpolation procedure.

Mode Shapes :

Eq. (3.4.2) gives,

$$\begin{aligned} -w_{NS} &= T_{13}\,M_o + T_{14}\,V_o \\ \psi_{NS} &= T_{23}\,M_o + T_{24}\,V_o \end{aligned} \qquad (3.4.5)$$

Assuming deflection at free end as unity, using Eqs. (3.4.3) and (3.4.5) the following relations can be obtained.

State Vector at Free End :

$$\begin{aligned} -w_{NS} &= 1.0 \\ \psi_{NS} &= (T_{23}T_{34} - T_{24}T_{33})/(T_{13}T_{34} - T_{14}T_{33}) \\ M_{NS} &= 0 \\ V_{NS} &= 0 \end{aligned} \qquad (3.4.6)$$

State Vector at Fixed End :

$$-w_o = 0$$

$$\psi_o = 0$$
$$M_o = T_{34}/(T_{13}T_{34} - T_{14}T_{33})$$
$$V_o = -T_{33}/(T_{13}T_{34} - T_{14}T_{33}) \qquad (3.4.7)$$

This initial state vector, premultiplied by successive local transfer matrices, would lead to state vectors at the ends of successive segments. So, the values of deflection at each segment end would lead to the mode shape. This procedure is applied to all the natural frequencies to get the crossponding mode shapes. The requirement that $- w_{NS} = 1.0$ at the tip provides a check an the mode shape computations.

3.5 Root Flexibility Effect :

In a turbomachinery the blades do not form an integral part of the disk, but are rather loosely mounted on the rotor disk. The vibrations of the blade, therefore, continue to some extent into the mounting. This has an effect similar to that of increased blade length and results in a lower frequency than the one calculated on the fixed end assumption. The root flexibility varies with the speed of rotation, since the centrifugal forces tend to tighten the blade in the disk and increase the fixidity.

The root flexibility, generally makes the root a non-deflecting end, but not a non-rotating end ; therefore, imposing boundary conditions

$$M_{NS} = V_{NS} = 0 \ ; \quad w_o = 0 \ , \ \psi_o = (\ M/K\ )_o$$

in Eq. (3.4.1) gives

$$\begin{bmatrix} -w \\ \psi \\ 0 \\ 0 \end{bmatrix}_{NS} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \begin{bmatrix} 0 \\ M/K \\ M \\ V \end{bmatrix}_{o} \tag{3.5.1}$$

where K is rolling stiffness factor. Obviously very high value of K leads to fixed end condition, whereas very small value lead to simply supported end.

The above equation leads to the following condition :

$$\begin{aligned} (T_{32}/K + T_{33})\, M_o + T_{34}\, V_o &= 0 \\ (T_{42}/K + T_{43})\, M_o + T_{44}\, V_o &= 0 \end{aligned} \tag{3.5.2}$$

Non-trivial solution of the above equations exist only when the corresponding determinant vanishes.

Therefore,

$$\begin{vmatrix} (\, T_{32}/K + T_{33}\,) & T_{34} \\ (\, T_{42}/K + T_{43}\,) & T_{44} \end{vmatrix} = 0 \tag{3.5.3}$$

The values of 'ω' for which the Eq. (3.5.3), called frequency equation, is satisfied are the natural frequencies of the system.

Mode Shapes :

Eq. (3.5.1) gives,

$$\begin{aligned} -w_{NS} &= (\, T_{12}/K + T_{13}\,)\, M_o + T_{14}\, V_o \\ \psi_{NS} &= (\, T_{22}/K + T_{23}\,)\, M_o + T_{24}\, V_o \end{aligned} \tag{3.5.4}$$

Assuming deflection at free end as unity, using Eqs. (3.5.2) and (3.5.4) the following relations can be obtained.

State Vector at Free End :

$$- w_{NS} = 1.0$$

$$\psi_{NS} = [(T_{22}T_{34} - T_{24}T_{32}) + K (T_{23}T_{34} - T_{24}T_{33})]/\tau$$

$$M_{NS} = 0$$

$$V_{NS} = 0 \qquad (3.5.5)$$

State Vector at Fixed End :

$$- w_o = 0$$

$$\psi_o = T_{34}/\tau$$

$$M_o = K\, T_{34}/\tau$$

$$V_o = - (T_{32} + K\, T_{33}) / \tau \qquad (3.5.6)$$

where

$$\tau == (T_{12}\, T_{34} - T_{14}\, T_{32}) + K (T_{13}\, T_{34} - T_{14}\, T_{33})$$

Effects of root flexibility can easily be taken into account by taking various values of K, such as 10^4, 10^{-4}, 10^{-8} etc. covering the whole range from fixed root condition to simply supported root condition.

CHAPTER IV

4.1 Formulation of State Matrix for Combined Bending and Torsion

The deflections at the centre of mass and the shear centre are connected by the equation.

$$v(x) = w(x) + e \cdot \beta(x) \qquad (4.1.1)$$

where $v(x)$ = deflection at the centre of mass of beam cross-section at distance 'x' from the root, $w(x)$ = deflection at shear center, $\beta(x)$ = torsional deflection, and e = distance between center of mass and shear center.

Considering force equilibrium for stagger angle 'θ', we have

$$\frac{d}{dx}V(x)\ dx + \mu(x)\omega^2\ v(x)\ dx + \mu(x)\Omega^2\ v(x)\,\text{Sin}^2\theta\ dx = 0$$

$$(4.1.2)$$

Substituting Eq. (4.1.1) into Eq. (4.1.2),

$$\frac{d}{dx}v(x) = -\mu(x)\ (\omega^2 + \Omega^2\text{Sin}^2\theta\ ).w(x) - \mu(x)(\omega^2 + \Omega^2\text{Sin}^2\theta\ )\ e.\beta(x)$$

$$(4.1.3)$$

Equilibrium of torsional moment gives

$$M_T - \left(\frac{d}{dx}M_T\ dx + M_T\right) + \mu(x)\omega^2\ v(x).e + \mu(x)\omega^2 i_y^2(x).\beta(x) = 0$$

$$(4.1.4)$$

or

$$\frac{d}{dx}M_T(x) = \mu(x)\ e\omega^2 .\ w(x) + \mu(x)\omega^2\ [\ i_y^2(x) + e^2]\ .\beta(x)$$

$$(4.1.4)$$

Consider the following equation :

$$\mu(x)\ [\ i_y^2(x) + e^2\ ] = \mu(x)\ I_p(x)/A(x)$$

$$(4.1.5)$$

where $I_p(x)$ = polar moment of inertia and $A(x)$ = area of cross-section.

Substituting Eq. (4.1.5) in Eq. (4.1.4), we get

$$\frac{d}{dx} M_T(x) = \mu(x)\omega^2 \; e. \; w(x) + \mu(x)\omega^2 \; I_p(x)/A(x).\beta(x) \tag{4.1.6}$$

Flexural moment equilibrium leads to the following equation,

$$\frac{dM(x)}{dx} = V(x) - \mu(x) \; i_y^2(x)\omega^2.\psi(x) - R(x). \frac{d}{dx}\left[\; w(x) + e.\beta(x) \; \right] \tag{4.1.7}$$

Slope 'ψ' is given by

$$\frac{dw(x)}{dx} = -\psi(x) + 1/GA_s(x) \; . \; V(x) \tag{4.1.8}$$

Simple Bending Moment theory gives the following relation,

$$\frac{d\psi(x)}{dx} = 1/EI(x). \; M(x) \tag{4.1.9}$$

Torsional deflection 'β' is related to the torsional moment 'M_T' by the following equation,

$$\frac{d\beta(x)}{dx} = 1/GJ(x).M_T(x) \tag{4.1.10}$$

where 'J' is mass moment of inertia about centre of mass.

Rewriting coupled bending-torsion equations,

$$\frac{dw(x)}{dx} = -\psi(x) + 1/GA_s(x) \; . \; V(x)$$

$$\frac{d\psi(x)}{dx} = 1/EI(x) \; . \; M(x)$$

$$\frac{dM(x)}{dx} = \left[\; R(x) - \mu(x) i_y^2(x)\omega^2 \; \right] .\psi(x) + V(x). \left[\; 1-R(x)/GA_s(x) \; \right] - R(x) e . 1/GJ(x).M_T(x)$$

$$\frac{dV(x)}{dx} = -\mu(x)\ (\omega^2 + \Omega^2 \sin^2\theta)\, .w(x)\ -\ \mu(x)\ (\omega^2 + \Omega^2 \sin^2\theta)\, e.\beta(x)$$

$$\frac{d\,M_T(x)}{dx} = \mu(x)\omega^2\ e.\ w(x)\ +\ \mu(x)\omega^2\ \frac{I_p(x)}{A(x)}\ \cdot\ \beta(x)$$

$$\frac{d\beta(x)}{dx} = 1/GJ(x).\ M_T \qquad (4.1.11)$$

Writing the state equations (4.1.11) in the matrix form:

$$\frac{d}{dx}\begin{bmatrix} -w(x) \\ \psi(x) \\ M(x) \\ V(x) \\ M_T(x) \\ \beta(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & Q(x) & 0 & 0 \\ 0 & 0 & P(x) & 0 & 0 & 0 \\ 0 & RN(x) & 0 & RM(x) & AT(x) & 0 \\ AM(x) & 0 & 0 & 0 & 0 & AU(x) \\ AS(x) & 0 & 0 & 00 & 0 & AV(x) \\ 0 & 0 & 0 & 00 & AW(x) & 0 \end{bmatrix} \begin{bmatrix} -w(x) \\ \psi(x) \\ M(x) \\ V(x) \\ M_T(x) \\ \beta(x) \end{bmatrix} \qquad (4.1.12)$$

where

$$R(x) = \int_x^L \mu(x)\,\Omega^2\ (x+r)\ dx$$

$$AM(x) = \mu(x)\ (\omega^2 + \Omega^2\ \sin^2\theta)$$

$$RN(x) = -\mu(x)\ i_y^2(x)\omega^2\ +\ R(x)$$

$$RM(x) = [\ 1 + R(x)\ .Q\ (x)\ ]$$

$$P(x) = 1/EI(x)$$

$$Q(x) = -1/GA_s(x)$$

$$AS(x) = -\ \mu(x)\,\omega^2\ \ e$$

$$AT(x) = -\ R(x)\ \cdot \frac{e}{GJ(x)}$$

$$AU(x) = -AM(x)\ .\ \ e$$

$$AV(x) = \mu(x)\,\omega^2\ \ I_p(x)\ /A(x)$$

$$AW(x) = \frac{1}{GJ(x)}$$

or

$$\frac{d}{dx}[z(x)] = [C(x)]\,[z(x)] \tag{4.1.13}$$

where $[z(x)]$ is the state vector and $[C(x)]$ is the state matrix of variable coefficients of first-order coupled bending-torsion differential equations (4.1.11).

The Runge-Kutta Method is again used to obtain the transfer matrix, which will be of the order 6 X 6.

4.2 Determination of Natural Frequencies and Mode Shapes

Writing the transfer matrix in the expanded form,

$$\begin{bmatrix} -w \\ \psi \\ M \\ V \\ M_T \\ \beta \end{bmatrix}_{NS} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} \\ T_{21} & T_{22} & T_{23} & T_{24} & T_{25} & T_{26} \\ T_{31} & T_{32} & T_{33} & T_{34} & T_{35} & T_{36} \\ T_{41} & T_{42} & T_{43} & T_{44} & T_{45} & T_{46} \\ T_{51} & T_{52} & T_{53} & T_{54} & T_{55} & T_{56} \\ T_{61} & T_{62} & T_{63} & T_{64} & T_{65} & T_{66} \end{bmatrix} \begin{bmatrix} -w \\ \psi \\ M \\ V \\ M_T \\ \beta \end{bmatrix}_{o} \tag{4.2.1}$$

Taking the root flexibility into account and imposing the boundary conditions

$$M_{NS} = V_{NS} = M_{T_{NS}} = 0 \;;\; w_o = \beta_o = 0\,,\; \psi_o = M_o/K_o$$

in Eq. (4.2.1)

$$\begin{Bmatrix} -w \\ \psi \\ 0 \\ 0 \\ 0 \\ \beta \end{Bmatrix}_{NS} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} \\ T_{21} & T_{22} & T_{23} & T_{24} & T_{25} & T_{26} \\ T_{31} & T_{32} & T_{33} & T_{34} & T_{35} & T_{36} \\ T_{41} & T_{42} & T_{43} & T_{44} & T_{45} & T_{46} \\ T_{51} & T_{52} & T_{53} & T_{54} & T_{55} & T_{56} \\ T_{61} & T_{62} & T_{63} & T_{64} & T_{65} & T_{66} \end{bmatrix} \begin{Bmatrix} 0 \\ M/K \\ M \\ V \\ M_T \\ 0 \end{Bmatrix}_{0} \qquad (4.2.2)$$

The above equation leads to the following condition :

$$(T_{32}/K + T_{33}) M_o + T_{34} V_o + T_{35} M_{T_o} = 0$$
$$(T_{42}/K + T_{43}) M_o + T_{44} V_o + T_{45} M_{T_o} = 0$$
$$(T_{52}/K + T_{53}) M_o + T_{54} V_o + T_{55} M_{T_o} = 0 \qquad (4.2.3)$$

Non-trivial solution of the above equations exist only when the corresponding determinant vanishes.

Therefore,

$$\begin{vmatrix} (T_{32}/K + T_{33}) & T_{34} & T_{35} \\ (T_{42}/K + T_{43}) & T_{44} & T_{45} \\ (T_{52}/K + T_{53}) & T_{54} & T_{55} \end{vmatrix} = 0 \qquad (4.2.4)$$

The values of 'ω' for which the Eq. (4.2.4), called frequency equation, is satisfied are the natural frequencies of the coupled bending- torsion system.

Mode Shapes :

Eq. (4.2.2) gives,

$$-w_{NS} = (T_{12}/K + T_{13})\,M_o + T_{14}\,V_o + T_{15}\,M_{T_o}$$

$$\psi_{NS} = (T_{22}/K + T_{23})\,M_o + T_{24}\,V_o + T_{25}\,M_{T_o}$$

$$\beta_{NS} = (T_{62}/K + T_{63})\,M_o + T_{64}\,V_o + T_{65}\,M_{T_o} \qquad (4.2.5)$$

Assuming deflection at free end as unity, using Eqs. (4.2.3) and (4.2.5) the following relations can be obtained.

State Vector at Free End :

$$\begin{aligned}
-w_{NS} &= 1.0 \\
\psi_{NS} &= A_2/B_5 \cdot T_{55} + B_2/B_6 \cdot T_{55}T_{24} - (A_5/B_5 + B_2/B_6 \cdot T_{54}) \cdot T_{25} \\
M_{NS} &= 0 \\
V_{NS} &= 0 \\
M_{T_{NS}} &= 0 \\
\beta_{NS} &= A_6/B_5 \cdot T_{55} + B_2/B_6 \cdot T_{55}T_{64} - (A_5/B_5 + B_2/B_6 \cdot T_{54}) \cdot T_{65}
\end{aligned} \qquad (4.2.6)$$

State Vector at Fixed End :

$$\begin{aligned}
-w_o &= 0 \\
\psi_o &= 1/B_5 \cdot K \quad T_{55} \\
M_o &= 1/B_5 \quad T_{55} \\
V_o &= B_2/B_6 \quad T_{55} \\
M_{T_o} &= -(A_5/B_5 + B_2/B_6 \cdot T_{54}) \\
\beta_o &= 0
\end{aligned}$$

where

$$A_1 = (T_{12}/K + T_{13})$$

$$A_2 = (T_{22}/K + T_{23})$$

$$A_3 = (T_{32}/K + T_{33})$$

$$A_4 = (T_{42}/K + T_{43})$$

$$A_5 = (T_{52}/K + T_{53})$$

$$A_6 = (T_{62}/K + T_{63})$$

$$B_1 = A_1 T_{55} - A_5 T_{15}$$

$$B_2 = A_5 T_{35} - A_3 T_{55}$$

$$B_3 = T_{14} T_{55} - T_{15} T_{54}$$

$$B_4 = T_{34} T_{55} - T_{35} T_{54}$$

$$B_5 = B_1 + B_2 \cdot B_3/B_4$$

$$B_6 = B_1 B_4 + B_2$$

CHAPTER V

5.1 RESULTS AND CONCLUSIONS

In this study transfer matrix approach is used to obtain the natural frequencies and mode shapes of large aspect ratio blades. The effect of the parameters like taper, root flexibility are also studied. The effect of shear deformation and rotatory inertia on the natural frequencies, which are of secondary importance are also investigated.

5.2. Specifications of the Blade

Length of blade	L	= 50.0 cm
Depth of blade at root	d	= 5.0 cm
Width of blade at root	b	= 1.3 cm
Radius of the disk at the blade root	r	=12.5 cm
Stagger angle	θ	= 1.3963
Young's modulus	E	= 21×10^8 gm/cm^2
Modulus of rigidity	G	= 8.3×10^8 gm/cm^2
Mass per unit length	μ	= 0.0516 gm/cm
Moment of inertia		
-- about z axis		= 13.54 cm^4
-- about y axis		= 0.9151 cm^4

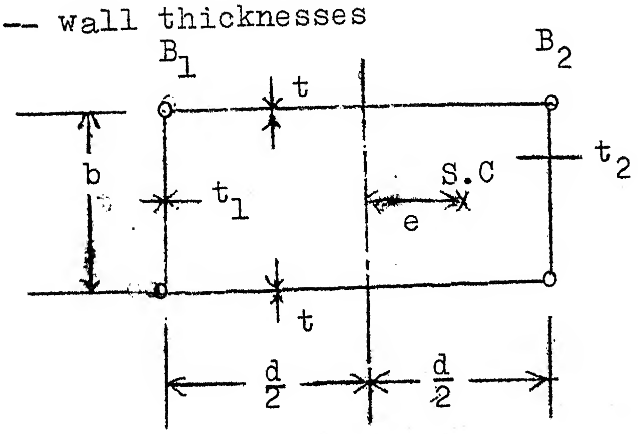
5.2.1 Determination of the Position of Shear Centre

The position of shear centre e, from the centroid of the beam of singly symmetric rectangular tubular crosssection as shown in figure below, is given by[φ]

$$e = \frac{d(B_2 - B_1)}{2(B_2 + B_1)} - \frac{2b(B_2/t_2 - B_1/t_1)}{(B_2+B_1)(\frac{b}{t_1} + \frac{b}{t_2} - \frac{2d}{t})}$$

where B_1, B_2 -- lumped areas at the corner of beam X-section

t, t_1, t_2 -- wall thicknesses



The shear centre positions for t = o.4 cm, t_1 = o.4 cm, d = 5 cm b = 1.3 cm and varying values of t_2 are listed below

S.No.	t_2 cm	e cm	Off-Set distance in o/o age of Chord ~~(approx.)~~
1+	0.40	0.0000	0 o/o
2	0.45	0.0773	2 o/o
3+	0.50	0.1444	3 o/o
4	0.55	0.2037	4 o/o
5+	0.60	0.2504	5 o/o
6	0.65	0.3055	6 o/o
7+	0.70	0.3499	7 o/o
8	0.75	0.3911	8 o/o
9+	0.80	0.4295	9 o/o

Taking + values for computation purposes, we can evaluate the effect of off-set distances 'e' on the natural frequencies of uniform beam.

φ Ref. Aeronautical Hand Book Vol. I

5.3 Numerical Computation

Computer programs are written for the evaluation of natural frequencies for pure bending and coupled bending-torsion mode based on the formulation given in Ch. III and IV. Fortran IV language is used for the programs which are run on IBM 7044 computer. Flow chart is given in Appendix A.

Effect of various values of double taper ratio on the variation of frequencies of first four bending modes, taking different number of segments are obtained for three values of rotational speeds (5000, 10000 and 15000 RPM). These are presented in Table 1 a, b and c. Bending frequencies of uniform beam for different number of segments are also obtained and presented in Table 2, for various values of rotational speeds (5000, 10000 and 15000 RPM). Convergence studies have been made and plotted as in Figs. 5.1.1 a, b both for non-uniform and uniform beam in bending mode at 5000 RPM. It is observed that the rate of convergence is faster and decreases with increasing modes.

Bending modes (first four) of uniform beam at 3000 RPM are presented in Table 3 and plotted in Fig. 5.1.2.

First bending and bending-torsion mode (with 3 o/o off-set distance 'e') of uniform beam at 3000 RPM are given in Table 4. The effect of off-set distance is shown in Fig. 5.1.3.

It is observed from the figure that the displacements are close to the pure bending mode for distances close to the support. However, as we move away from the support, they start diverging with the maximum value occuring at the tip.

Bending mode (second) of uniform beam for different rotational speeds (3000,6000, and 10000 RPM) are given in Table 5. These results are plotted in Fig. 5.1.4, which shows that as the rotational speed increases say from 3000 RPM to 10000 RPM, the beam region between 20 o/o to 60 o/o is affected much more than that near the tip region. The reason being that tip deflection is constrained to unity for computational purposes.

Coupled bending-torsion mode of vibration of uniform beam is obtained for five sets of off-set distance (e). These are presented in Table 7, for 3000 RPM and plotted in Fig. 5.1.5.

The variation of first four bending natural frequencies with the change in the stagger angle (θ) of the root of the blade is shown in Table 6. An increase of θ from 0° to 90° decreases the fundamental frequency by about 14.5 o/o. The other frequencies arenot significantly affected. These are plotted in Fig. 5.1.5a.

The effect of root flexibilities on the frequencies of first four bending modes of non-uniform(double taper

ratio = 0.5) as well as that of uniform beam are presented in Tables 8 and 9 by taking various values of rolling stiffness factor K (eg. $10^{38}, 10^{20}, 10^{4}$ and 10^{-8}), for various values of rotational speeds (0,5000 and 10000 RPM). These results are plotted in Fig. 5.1.6 a and b, which show that as rolling stiffness factor is increased, there is a marked increase in the natural frequencies with the increase in the rotational speed. The results obtained for various values of K in the limit coincides with those obtained for a cantilever beam.

Bending frequencies of non-uniform (double taper ratio = 0.5) as well as uniform beam for various values of rotational speeds (from 0 RPM to 10000 RPM) are also obtained to show their effect and presented in Table 10.

The variation of frequency parameter ratio with depth and width taper are obtained for various values of rotational speeds (0,5000,10000 and 15000 RPM) and presented in Table 11.

The above results for the first three modes are plotted in Figs. 5.1.7 a,b and c and are compared with the results of Rao and Carnegie [91]. It can be seen that they are in good agreement,accounting for shear deformation and rotary inertia effects, for non-rotating case.

For the non-rotating case, for a given width taper the frequency parameter ratio for the fundamental mode decreases as depth taper increases. For the second and third

modes, the frequency parameter ratio shows a marked increase with the depth taper. Also for a given depth taper the frequency parameter ratio decreases as the width taper increases for all the three modes.

The effect of rotational speed is also shown in Fig. 5.1.7.

CHAPTER VI

RECOMMENDATIONS

The following are some of the points on which further investigations are recommended to be carried out.

1) Use of transfer matrix approach for obtaining natural frequencies of twisted, tapered and rotating blades with and without root flexibility.

2) The effect of disk elasticity.

3) The effect of camber on the natural frequencies.

4) Extension to small aspect ratio blades.

5) Study of Flutter characteristics.

Effect of taper ratio on the variation of frequency with no. of segments for various values of rotational speeds.

NON-UNIFORM BEAM (BENDING MODE)

Ω = 5000 RPM

DOUBLE TAPER RATIO = 0.25

NO OF SEGMENTS 6	10	12	20	25	30
953.410	952.50	948.141	945.286	930.795	
2160.173	2156.650	2138.207	2131.133	2098.588	
4046.542	4039.729	4017.409	4009.431	4028.257	
6710.714	6700.366	6695.980	6716.408	5613.935	

DOUBLE TAPER RATIO = 0.50

710.252	716.722	717.950	719.793	720.174	720.178
2119.824	2131.146	2134.814	2140.497	2141.769	2141.822
4424.219	4487.227	4484.998	4488.785	4490.707	4493.411
7980.466	7951.831	7916.789	7883.333	7882.302	7882.352

DOUBLE TAPER RATIO = 0.75

575.030	577.522	578.002	578.726	578.876
2175.341	2176.039	2176.953	2178.792	2179.241
5012.378	4947.745	4940.388	4935.905	4936.079
8934.273	8964.821	9016.160	8965.729	8960.384

TABLE 1 (a)

NON-UNIFORM BEAM (BENDING MODE)

Ω = 10000 RPM

DOUBLE TAPER RATIO = 0.25

No. OF SEGMENTS 6	10	12	20	25
1712.350	1711.765	1705.645	1702.528	1749.352
3525.684	3522.574	3491.852	3480.493	3508.060
5925.684	5903.824	5851.852	5463.291	5233.306
9525.684	9569.825	9601.852	9611.528	9643.306

DOUBLE TAPER RATIO = 0.50

1269.823	1271.016	1272.827	1275.778	1276.404
3358.786	3357.414	3362.191	3370.789	3373.193
6243.401	6179.518	6177.232	6175.055	6181.053
8962.151	9822.376	9897.232	9898.009	9866.386

DOUBLE TAPER RATIO = 0.75

998.840	999.707	1000.437	1001.594	1001.848
3337.857	3335.167	3336.145	3338.697	3339.243
6604.857	6530.366	6500.394	6516.417	6515.465
10526.667	10931.259	10881.932	10846.348	10835.710

TABLE 1 (b)

NON-UNIFORM BEAM (BENDING MODE)

Ω = 15,000 RPM

DOUBLE TAPER RATIO = 0.25

No. OF SEGMENTS				
6	10	12	20	25
2512.143	2511.233	2505.189	2503.008	2567.907
4325.645	4322.042	4291.644	4280.727	4308.060
6725.978	6703.744	6651.144	6263.958	6033.306
10325.978	10369.744	10401.989	10411.445	10443.308

DOUBLE TAPER RATIO = 0.50

1854.442	1857.407	1958.000	1861.429	1872.667
4655.742	4677.407	4693.750	4632.929	4797.071
8421.740	8539.407	8600.417	8632.928	8890.571
16645.740	15460.409	14960.417	13823.928	11142.904

DOUBLE TAPER RATIO = 0.75

1413.684	1434.187	1436.966	1439.583	1437.734
4632.449	4638.187	4651.739	4661.196	4659.347
8479.480	8445.245	8439.239	8427.862	8492.661
18624.480	18556.245	17202.863	13339.239	12955.181

TABLE 1 (c)

Variation of frequency with no. of segments for various values of rotational speeds.

UNIFORM BEAM (BENDING MODE)

Ω = 5000 RPM

No. Of Segments = 25

6	10	12	20	25	30
478.238	478.024	478.008	477.996	477.995	477.996
2238.627	2232.101	2231.498	2230.984	2230.934	2230.933
5481.433	5393.274	5382.948	5372.551	5371.541	5371.540
10045.142	10116.017	10053.230	9986.684	9980.766	9980.765
Ω = 10000 RPM					
798.656	797.518	797.433	797.386	797.388	
3345.969	3340.247	3339.418	3338.714	3338.719	
6969.754	6887.774	6876.328	6865.050	6864.230	
11692.021	11874.751	11809.415	11750.466	11744.502	
Ω = 15000 RPM					
1132.893	1124.049	1124.979	1123.354	1125.152	
4616.241	4609.635	4607.820	4606.975	4606.569	
8810.241	8779.655	8768.156	8759.975	8759.815	
13049.241	13083.635	14205.656	19509.976	21284.816	

TABLE 2

Bending Mode Shapes of Uniform Beam at 3000 RPM

MODE SHAPES

Interval End	I	II	III	IV
1	0.0000	0.0000	0.0000	0.0000
2	0.0037	-0.0156	0.0418	-0.0819
3	0.0143	-0.0568	0.1434	-0.2634
4	0.0307	-0.1163	0.2764	-0.4676
5	0.5233	-0.2639	0.4156	-0.6291
6	0.0785	-0.3402	0.5347	-0.7004
7	0.1086	-0.4109	0.6185	-0.6571
8	0.1422	-0.4715	0.6527	-0.5005
9	0.1788	-0.5179	0.6305	-0.2561
10	0.2182	-0.5469	0.5518	+0.0326
11	0.2578	-0.5561	0.4226	+0.3122
12	0.3034	-0.5438	0.2548	0.5305
13	0.3486	-0.5094	0.0644	0.6464
14	0.3953	-0.4529	-0.1296	0.6379
15	0.4432	-0.3854	-0.3076	0.5073
16	0.4921	-0.3785	-0.4508	0.2801
17	0.5417	-0.2418	-0.5438	0.0008
18	0.5920	-0.0865	-0.5753	-0.2753
19	0.6426	++0.0834	-0.5400	-0.4927
20	0.6936	0.2616	-0.4383	-0.6058
21	0.7447	0.4445	-0.2762	-0.5876
22	0.7959	0.6297	-0.0648	-0.4366
23	0.8470	0.8151	+0.1819	-0.1617
24.	0.8991	0.8542	+0.4494	+0.1935
25	0.9491	0.9005	+0.7247	+0.5918
26	1.0000	1.0000	1.0000	1.0000

TABLE -3-

Effect of Off-Set Distance (3o/o) on the first Bending Mode Shapes of Uniform Beam at 3000 RPM

MODE SHAPES

INTERVAL END	PURE BENDING	BENDING-TORSION
1	0.0000	0.0000
2	0.0049	0.0051
3	0.0193	0.0226
4	0.0428	0.0476
5	0.0749	0.0871
6	0.1156	0.1402
7	0.1629	0.2082
8	0.2172	0.2929
9	0.2774	0.3960
10	0.3422	0.5200
11	0.4106	0.6676
12	0.4814	0.8427
13	0.5530	1.0494
14	0.6342	1.2940
15	0.6936	1.5838
16	0.7593	1.9285
17	0.8202	2.3441
18	0.8748	2.8382
19	0.9218	3.4419
20	0.9601	4.1817
21	0.9889	5.0677
22	1.0080	6.2394
23	1.0173	7.6807
24	1.0177	9.5171
25	1.0110	11.8804
26	1.0000	1.0000

TABLE 4

Variation of Bending Mode Shapes (Second Mode) with Rotational Speeds.

MODE SHAPES

Interval End	3000 RPM	6000 RPM	10000
1	0.0000	0.0000	0.0000
2	-0.0156	-0.0178	-0.0199
3	-0.0568	-0.0630	-0.0671
4	-0.1163	-0.1269	-0.1301
5	-0.2639	-0.2024	-0.2013
6	-0.3402	-0.2834	-0.2753
7	-0.4109	-0.3647	-0.3478
8	-0.4715	-0.4419	-0.4155
9	-0.5179	-0.5107	-0.4753
10	-0.5469	-0.5678	-0.5248
11	-0.5561	-0.6099	-0.5615
12	-0.5438	-0.6346	-0.5833
13	-0.5094	-0.6395	-0.5882
14	-0.4592	-0.6232	-0.5746
15	-0.3854	-0.5847	-0.5412
16	-0.3785	-0.5233	-0.4869
17	-0.2418	-0.4396	-0.4113
18	-0.0865	-0.3342	-0.3145
19	+0.0800	-0.2089	-0.1972
20	0.2616	-0.0659	-0.0608
21	0.4445	0.0922	0.0923
22	0.6297	0.2662	0.2595
23	0.8151	0.4410	0.4374
24	0.8542	0.6253	0.6222
25	0.9005	0.8122	0.8106
26	1.0000	1.0000	1.0000

TABLE 5

Effect of Stagger Angle variation at 3000 RPM

NON - UNIFORM BEAM(BENDING MODE)

DOUBLE TAPER RATIO = 0.5
NO. of Segments = 25

$\theta = 0^\circ$	$\theta = 30^\circ$	$\theta = 60^\circ$	$\theta = 90^\circ$
605.377	584.557	540.786	517.353
1789.348	1781.412	1768.475	1761.445
4039.192	4036.148	4030.027	4026.949
7376.347	7375.046	7371.758	7369.035

UNIFORM BEAM (BENDING MODE)

477.067	450.500	392.018	359.120
1932.958	1926.594	1913.728	1907.243
4995.562	4993.061	4988.118	4985.626
9560.126	9559.249	9556.608	9554.290

TABLE 6

Effect of off-set distance on the variation of frequency

COUPLED BENDING - TORSION MODE

UNIFORM BEAM

Ω = 3000 RPM

No. of Segments = 25

OFF-SET DISTANCE IN PERCENTAGE OF CHORD				
0 o/o	3 o/o	5 o/o	7 o/o	9 o/o
453.568	453.583	546.718	547.800	547.823
2389.931	2390.000	2390.326	2390.369	2400.526
6320.158	6307.653	6279.447	6310.715	6331.715
12226.977	12175.150	12118.014	12243.515	12189.549

TABLE 7

Effect of Root Flexibility on frequency for various values of rotational speeds.

NON-UNIFORM BEAM (BENDING MODE)

DOUBLE TAPER RATIO = 0.5

Ω = 10000 RPM

ROLLING STIFFNESS FACTOR, K 10^{38}	10^{20}	10^{4}	10^{-8}
1276.404	1276.404	1173.944	1173.969
3373.193	3373.193	3141.766	3142.256
6181.053	6181.053	5726.058	5725.801
9866.386	9866.386	9184.712	9139.434
		Ω = 5000 RPM	
720.174	720.174	594.285	594.279
2141.769	2141.769	1817.652	1817.645
4490.707	4490.707	3872.718	3872.686
7882.302	7882.302	6968.774	6967.693
		Ω = 0 RPM	
356.129	356.129	236.514	236.528
1505.129	1505.129	1016.669	1016.646
3738.624	3738.624	2977.451	2977.432
7063.125	7063.125	6035.462	6035.404

TABLE 8

Effect of Root Flexibility on frequency for various values of rotational speeds.

UNIFORM BEAM (BENDING MODE)

Ω = 10000 RPM

ROLLING STIFFNESS FACTOR , K			
10^{38}	10^{20}	10^{4}	10^{-8}
797.388	797.388	663.999	664.003
3338.719	3338.719	3084.058	3084.087
6864.230	6864.230	6289.727	6289.638
11744.502	11744.502	10785.215	10784.680

Ω = 5000 RPM

477.995	477.995	332.387	332.381
2230.934	2230.934	1869.379	1869.369
5371.542	5371.542	4607.372	4607.360
9980.766	9980.766	8821.133	8821.199

Ω = 0 RPM

271.177	271.177	185.395	185.401
1698.810	1698.810	1189.734	1189.734
4753.240	4753.240	3854.822	3854.822
9306.508	9306.508	8037.185	8037.185

TABLE 9

Variation of frequency with the rotational speeds

NON-UNIFORM BEAM (BENDING MODE)

DOUBLE TAPER RATIO = 0.50

No. OF SEGMENT = 25

0 RPM	2000	4000	6000	8000	10000
356.129	500	750	1050	1250	1276.404
1505.129	1700	2250	2550	3150	3373.193
3738.624	4100	4350	4950	5550	6181.053
7063.125	7400	7650	8250	9150	9866.886

UNIFORM BEAM (BENDING MODE)

271.177	316.264	418.398	540	700	797.388
1698.810	1894.874	2055.796	2427.218	3100	3338.719
4753.240	4858.239	5158.660	6619.409	6400	6864.230
9306.508	9618.032	9743.013	10280.648	11200	11744.502

TABLE 10

Variation of frequency parameter ratio with depth and width taper.

Ω = 0 RPM

ω_o = 271.177 CPS

Mode	Width Taper, γ	Depth Taper, δ 0.25	0.50	0.75
1	0.25	2.733	2.402	2.240
	0.50	2.001	1.726	1.587
	0.75	1.634	1.383	1.264
2	0.25	27.415	35.131	42.927
	0.50	23.993	30.796	37.657
	0.75	22.244	28.566	34.919
3	0.25	136.703	201.466	268.464
	0.50	128.133	189.547	253.221
	0.75	123.754	183.578	245.734

TABLE 11 a

Ω = 5000 RPM

ω_o = 477.995 CPS

1	0.25	3.79	2.61	1.765
	0.50	3.52	2.28	1.515
	0.75	3.35	2.18	1.435
2	0.25	19.2	22.3	22.7
	0.50	17.2	20.998	21.5
	0.75	16.1	20.291	20.8
3	0.25	70.55	91.50	108.195
	0.50	66.50	88.50	105.762
	0.75	63.95	87.00	103.200

TABLE 11 b

Variation of frequency parameter ratio with depth and width taper.

Ω = 10000 RPM

ω_o = 797.388 CPS

Mode	Width Taper, γ	Depth Taper, δ 0.25	0.50	0.75
1	0.25	4.797	2.89	1.87
	0.50	4.797	2.55	1.665
	0.75	4.797	2.39	1.57
2	0.25	18.3	18.45	18.72
	0.50	18.3	17.82	17.90
	0.75	18.3	17.72	17.50
3	0.25	43	61.70	68.81
	0.50	43	60	67.32
	0.75	43	59.25	66.50

TABLE 11 c

Ω = 15000 RPM

ω_o = 1125.152 CPS

1	0.25	5.200	3.01	1.71
	0.50	5.21	2.77	1.65
	0.75	5.25	2.74	1.63
2	0.25	14.70	18.35	17.65
	0.50	14.30	18.20	17.32
	0.75	14.10	18.10	17.20
3	0.25	28.70	62.00	58.00
	0.50	27.60	61.50	57.38
	0.75	27.10	61.40	57.10

TABLE 11 d

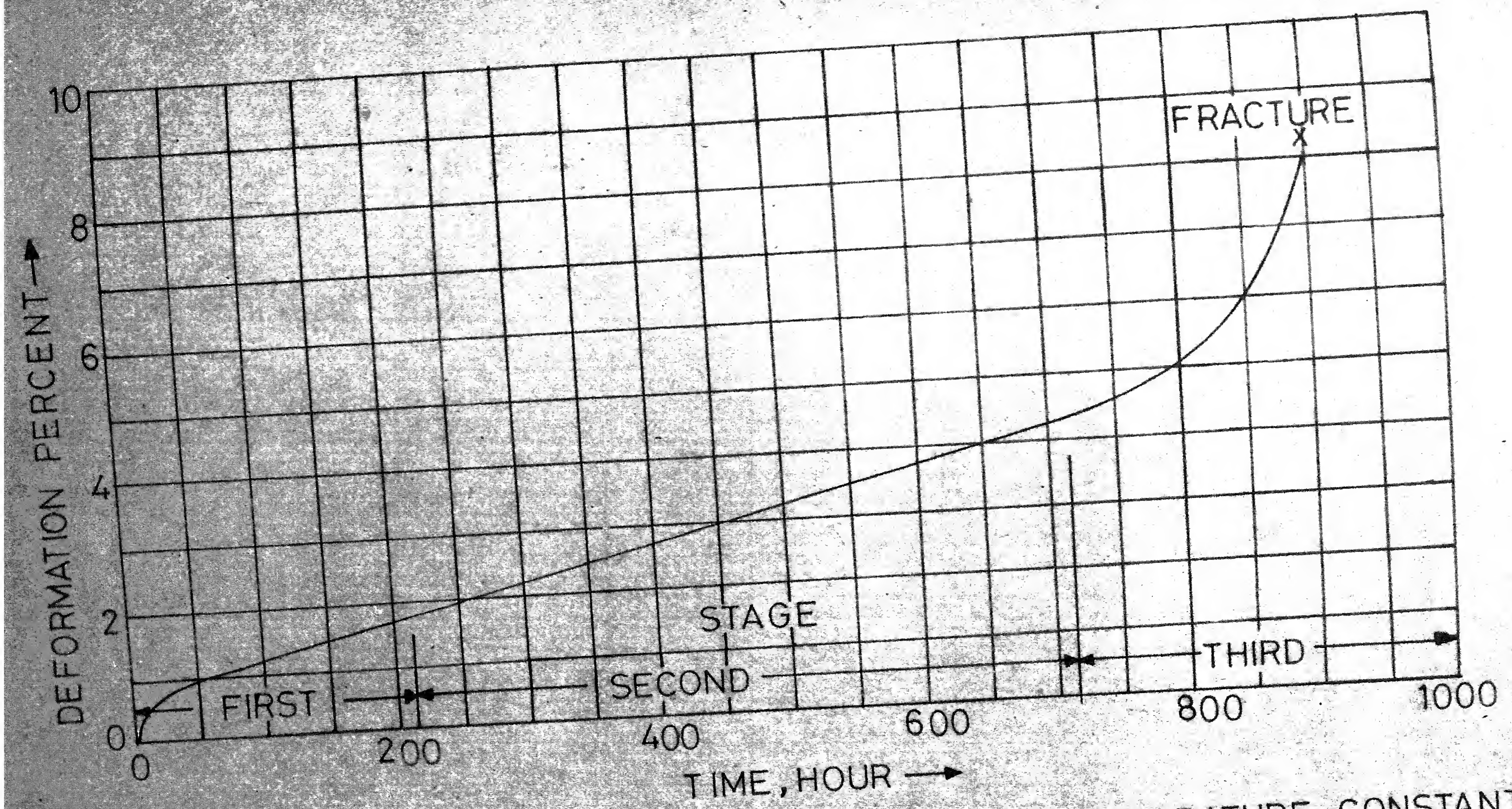


FIG. 2.11.2 - IDEAL CREEP CURVE (STRESS AND TEMPERATURE CONSTANT)

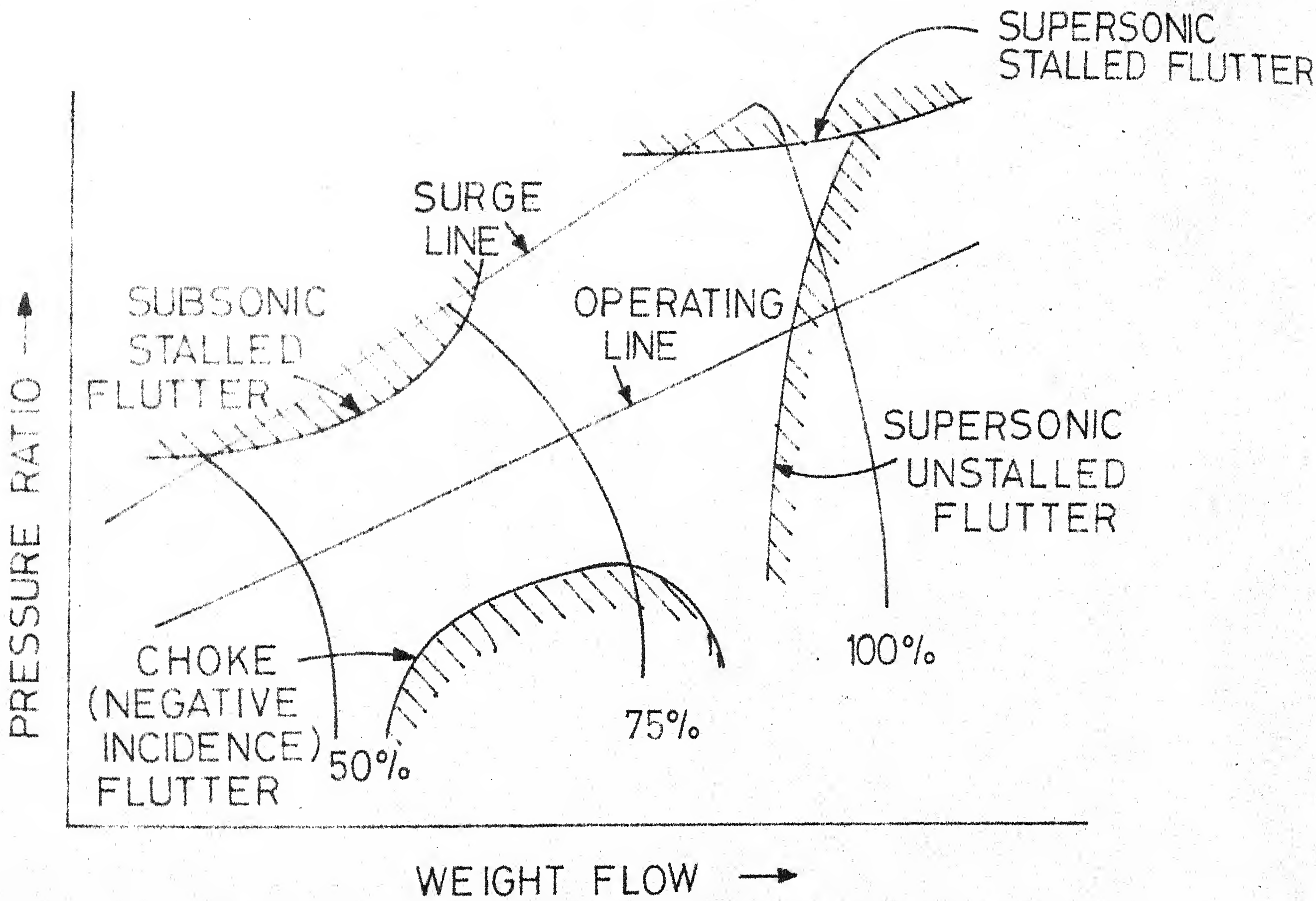


FIG. 2.13.1— COMPRESSOR MAP SHOWING BOUNDARIES FOR FOUR TYPES OF FLUTTER.

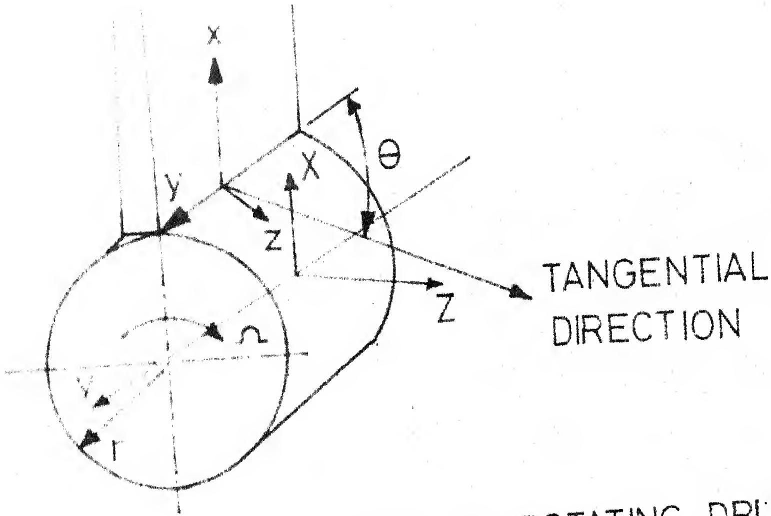


FIG. 3.2.1 a– BEAM MOUNTED ON ROTATING DRUM.

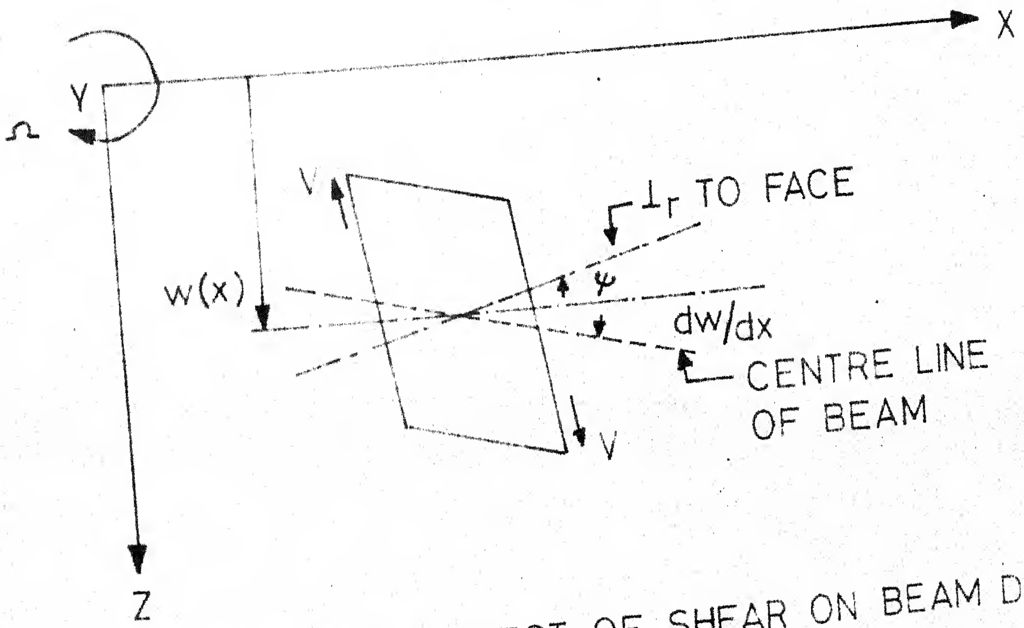


FIG. 3.2.1–b– THE EFFECT OF SHEAR ON BEAM DEFELECTION

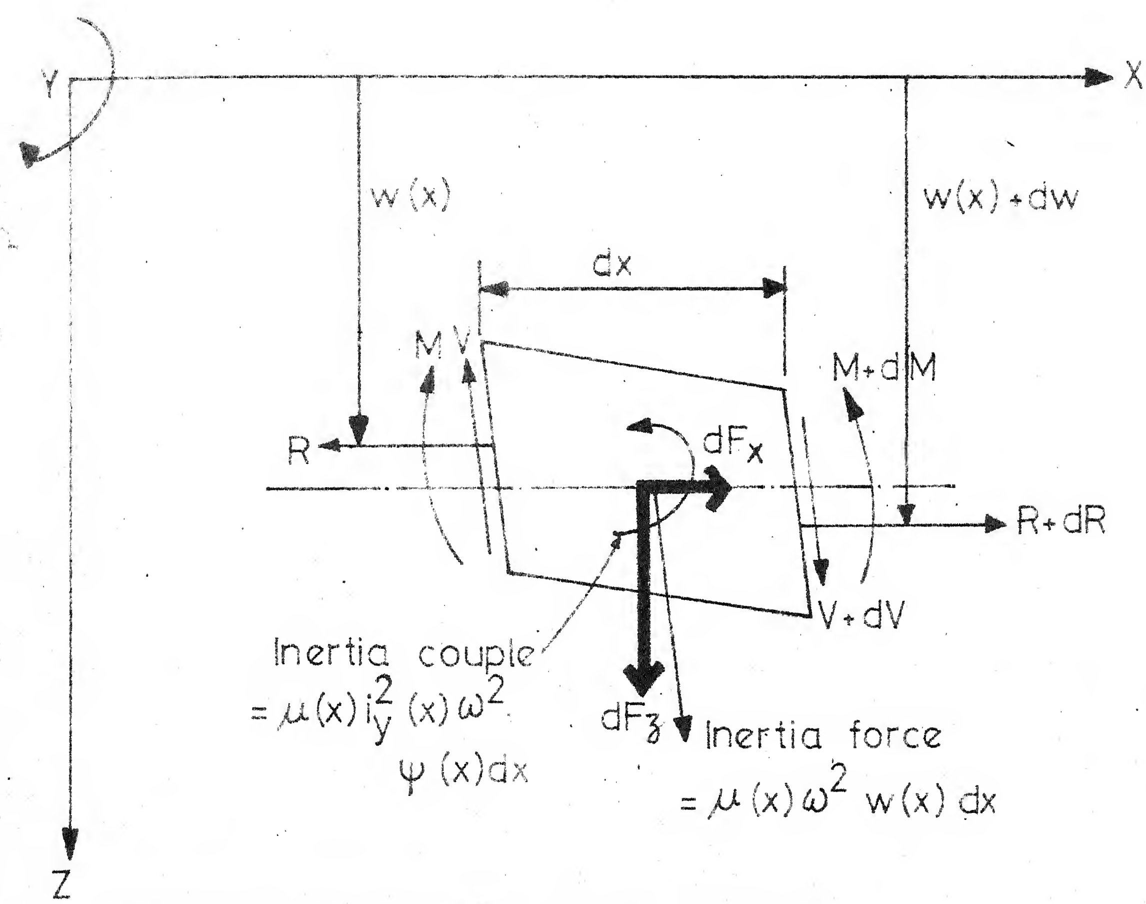


FIG. 3.2.2 - FORCE BALANCE DIAGRAM.

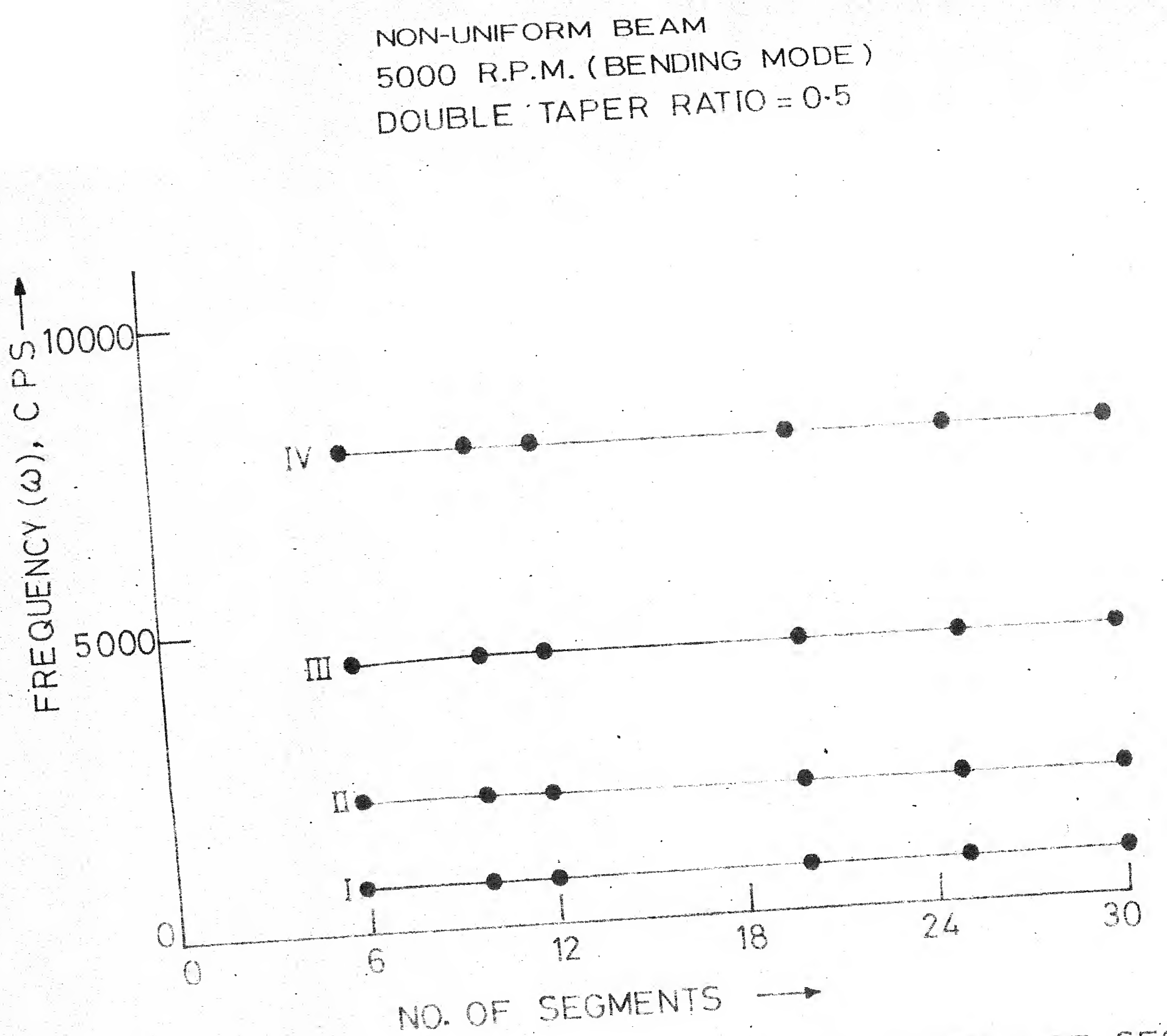


FIG.5·11-a– VARIATION OF FREQUENCY WITH NUMBER OF SEGMENTS.

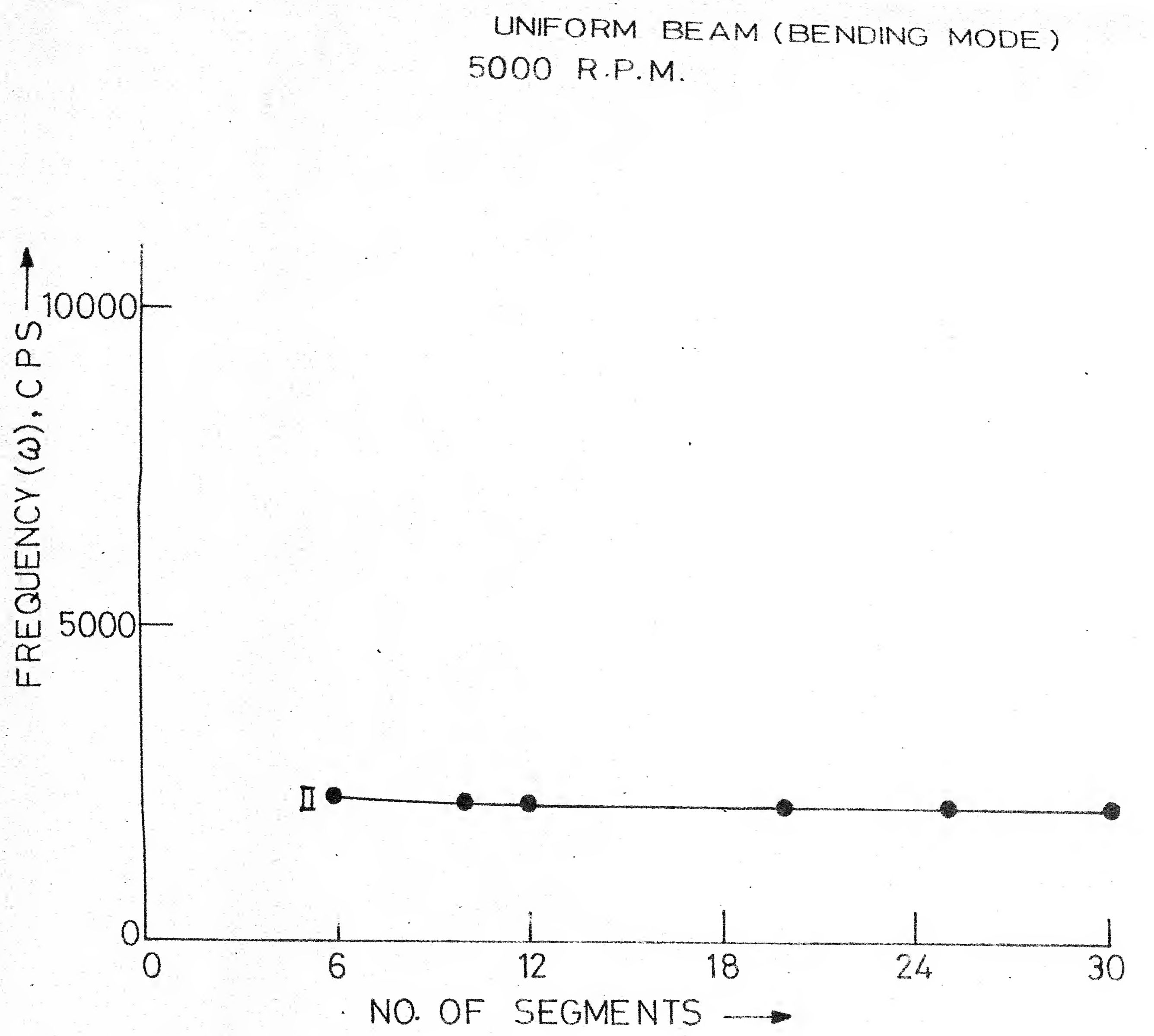


FIG. 5.1.1-b - VARIATION OF FREQUENCY WITH NUMBER OF SEGMENTS.

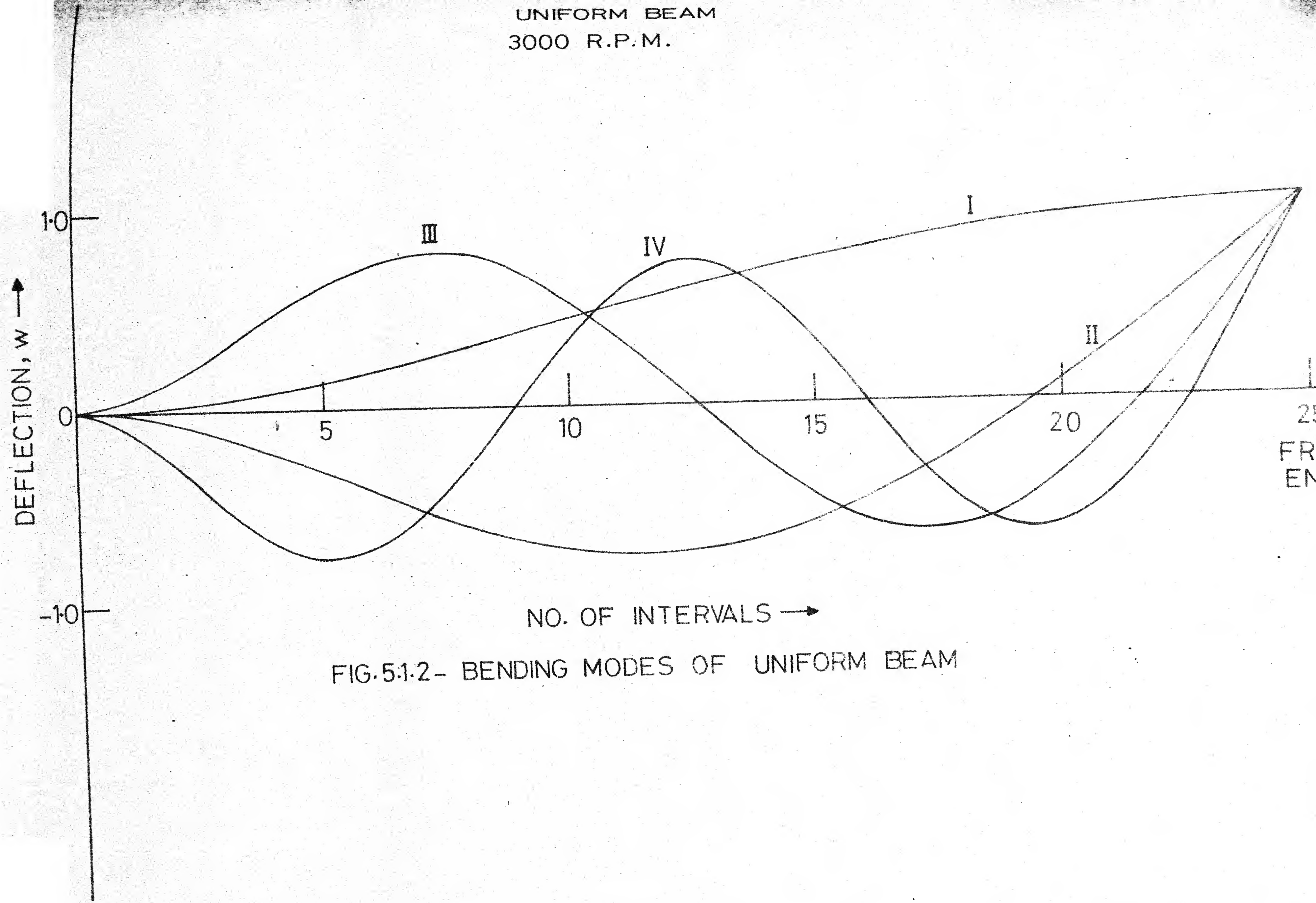


FIG. 5·1·2 – BENDING MODES OF UNIFORM BEAM

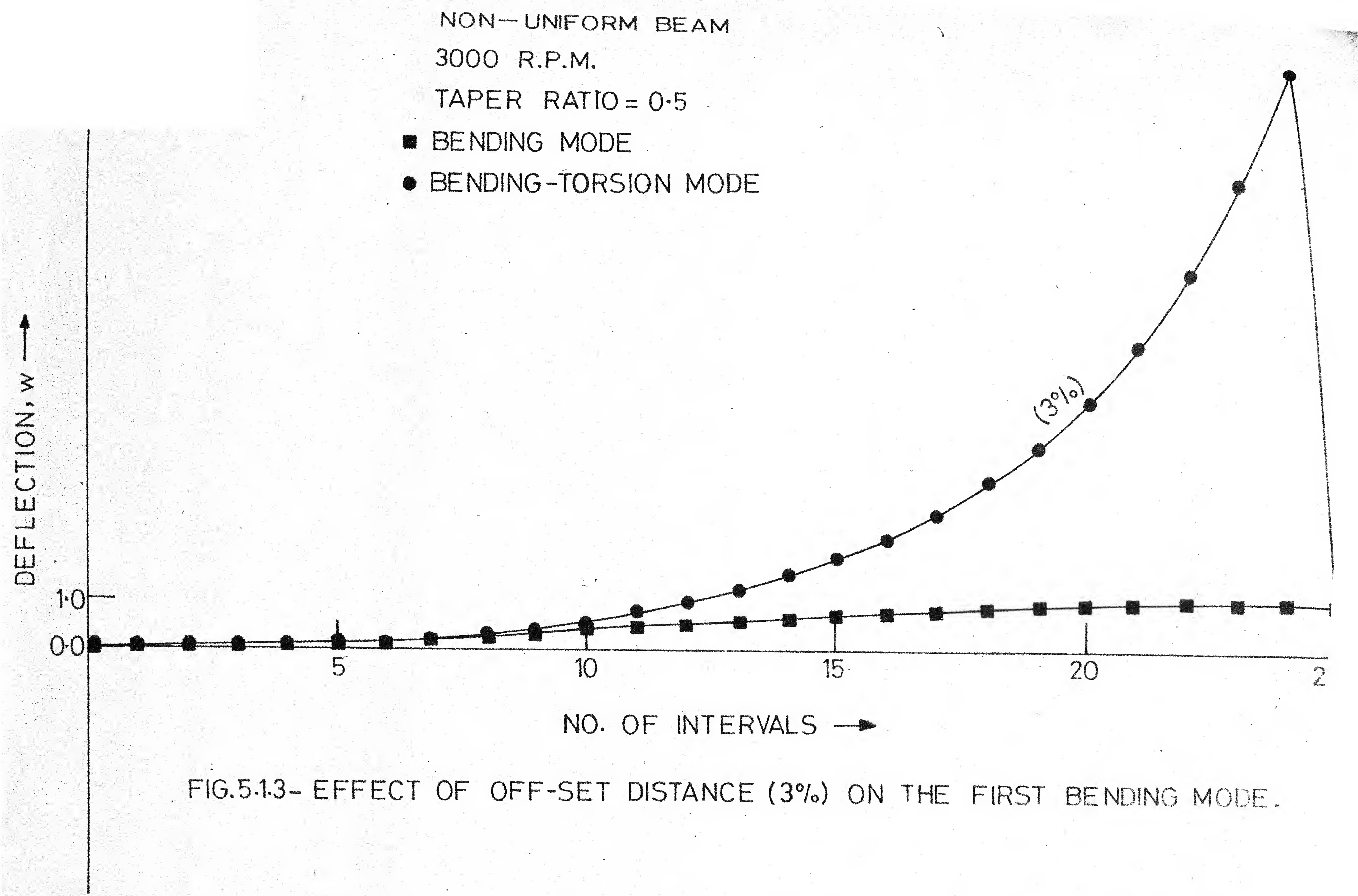


FIG.5.1.3- EFFECT OF OFF-SET DISTANCE (3%) ON THE FIRST BENDING MODE.

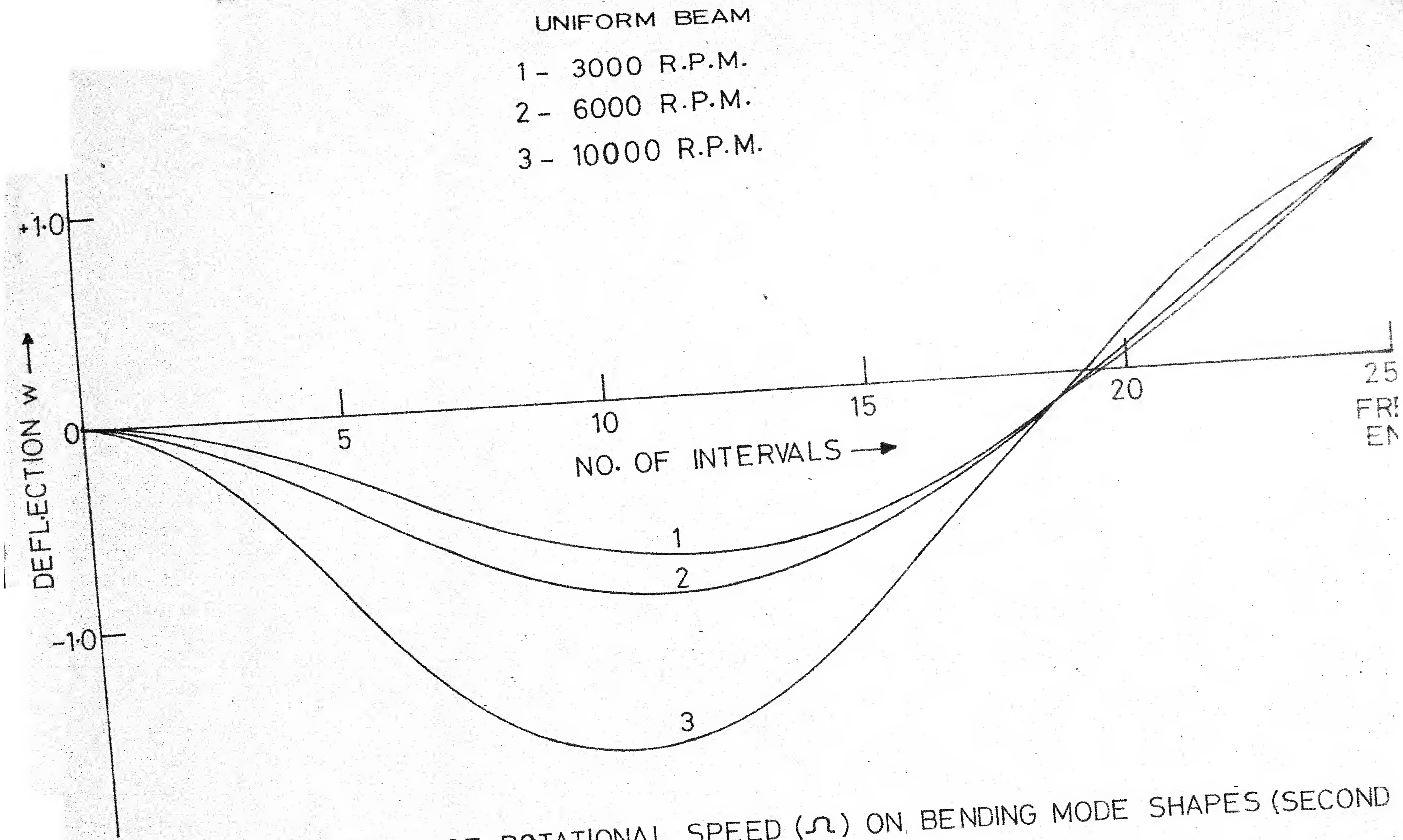


FIG.5.1.4 - EFFECT OF ROTATIONAL SPEED (Ω) ON BENDING MODE SHAPES (SECOND

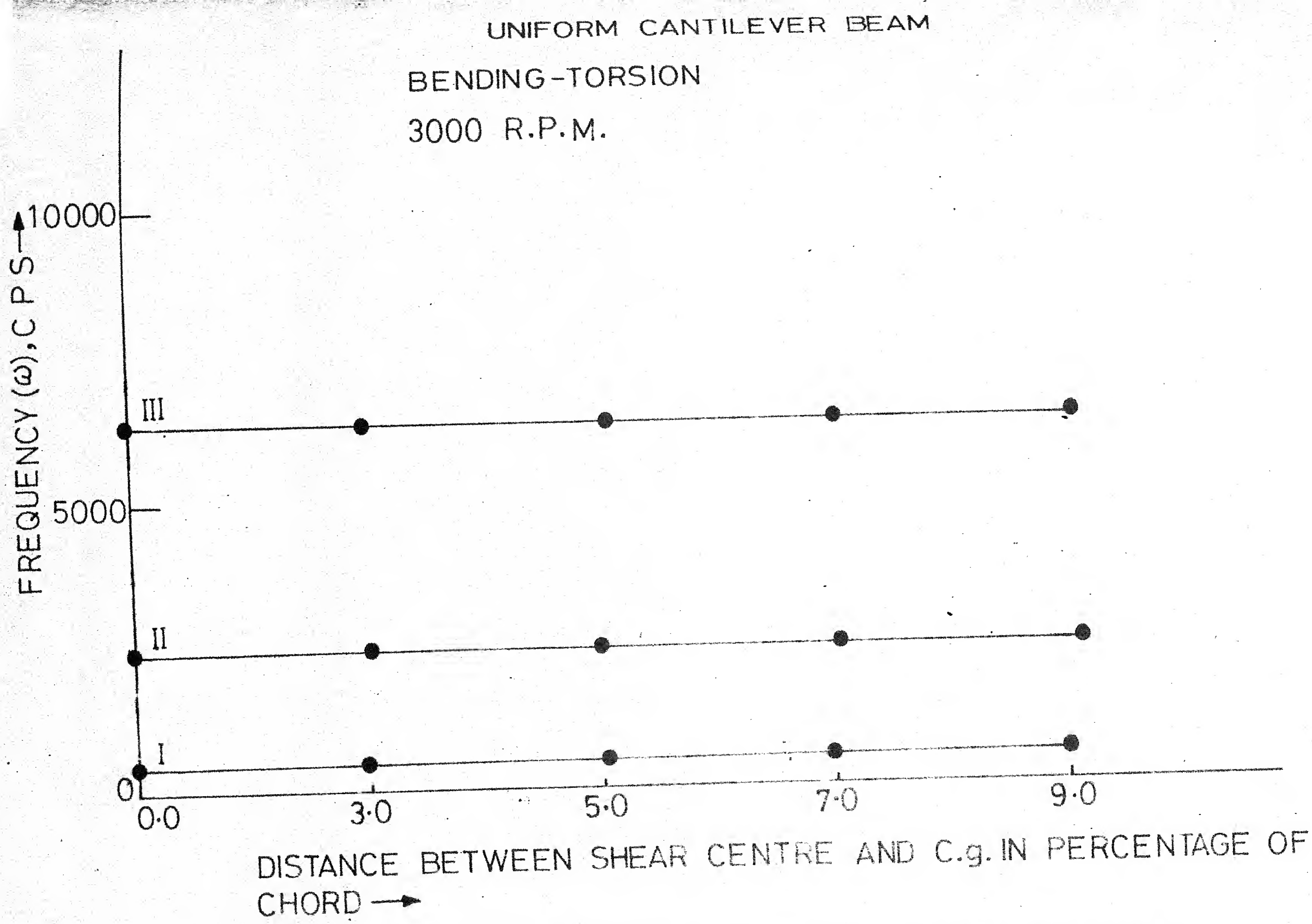


FIG. 5.1.5 – VARIATION OF FREQUENCY WITH OFF-SET DISTANCE.

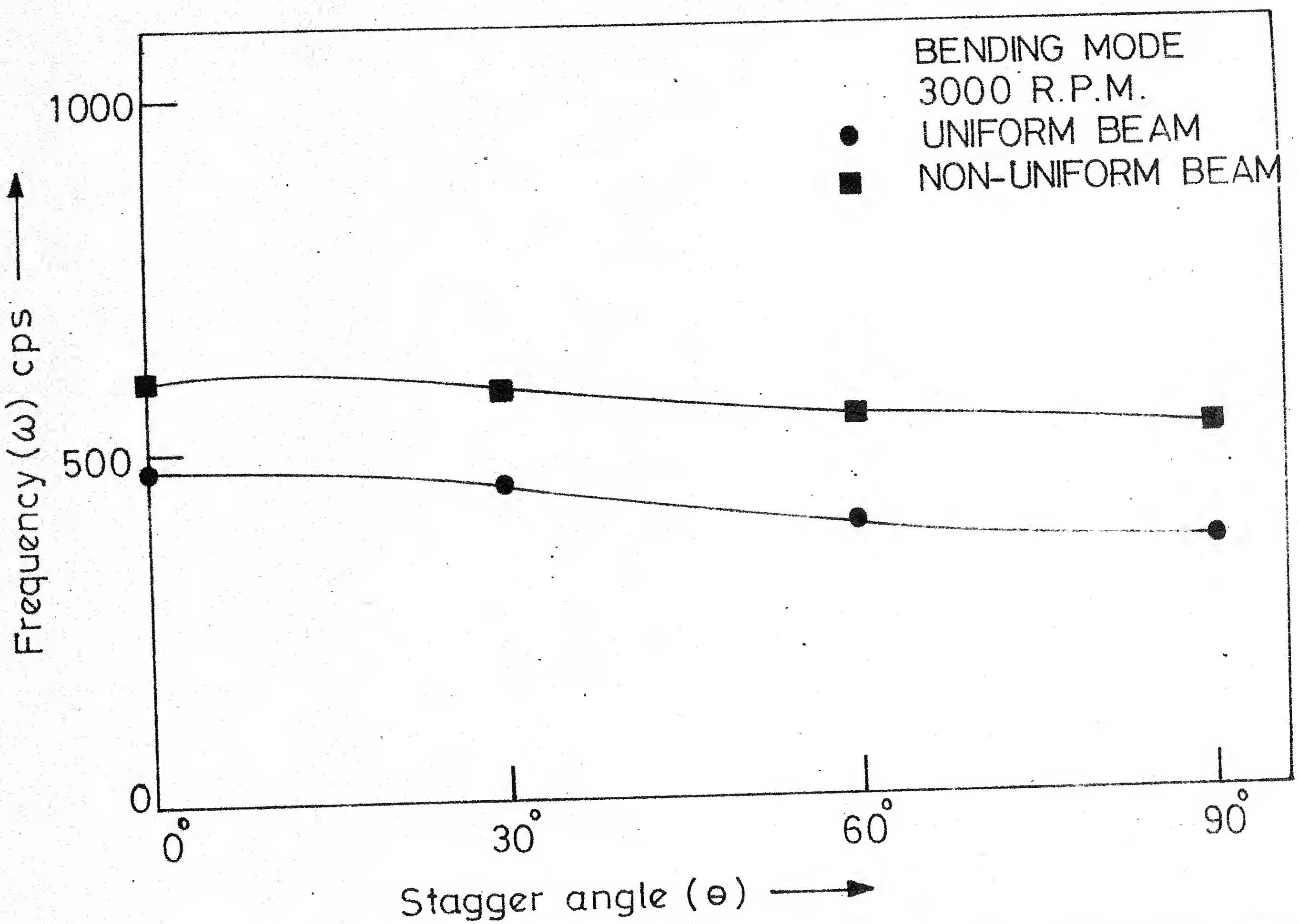


FIG. 5.1.5-a-VARIATION OF FREQUENCY WITH STAGGER ANGLE.

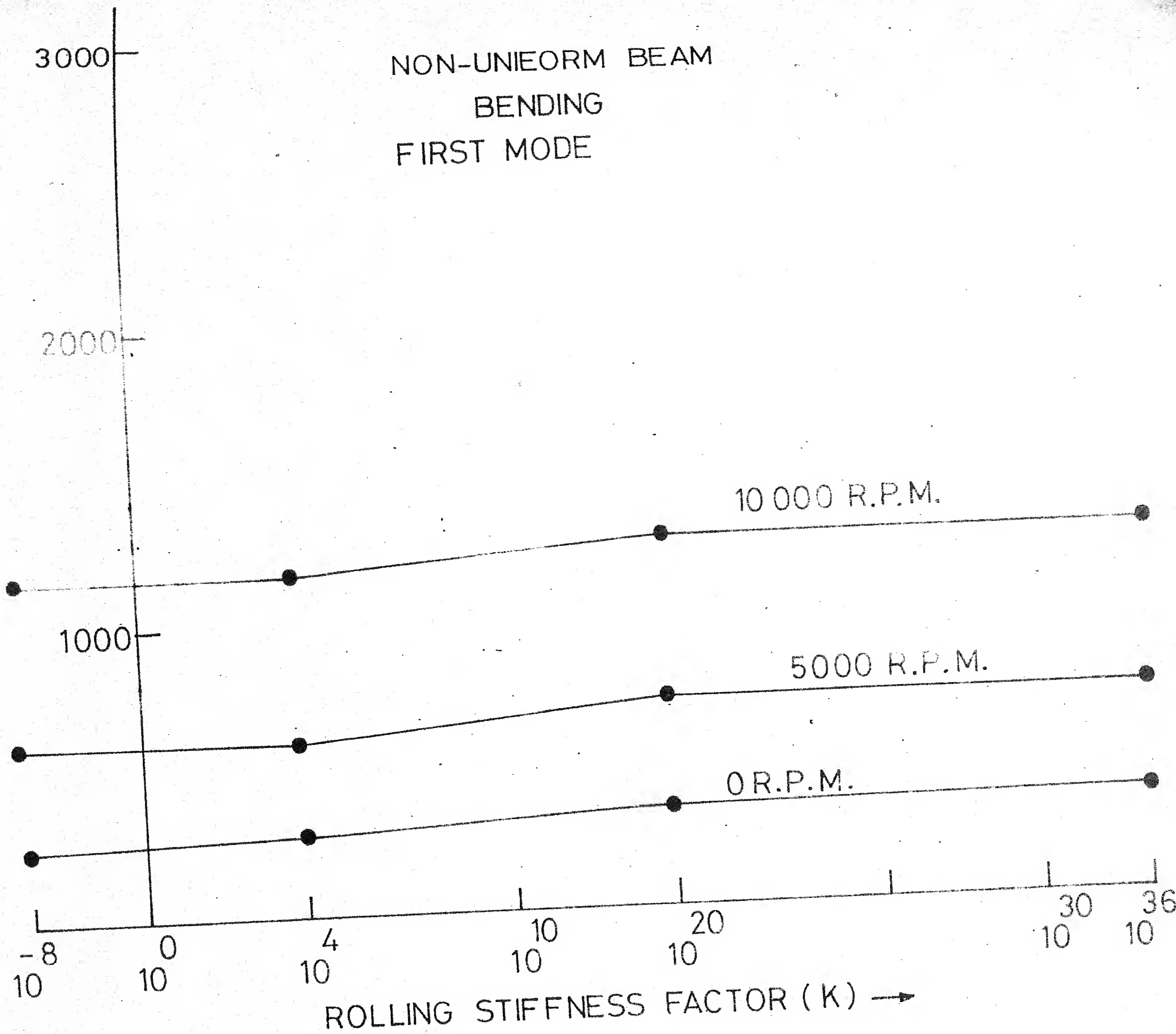


FIG. 5.1.6-a- EFFECT OF ROOT FLEXIBILITY.

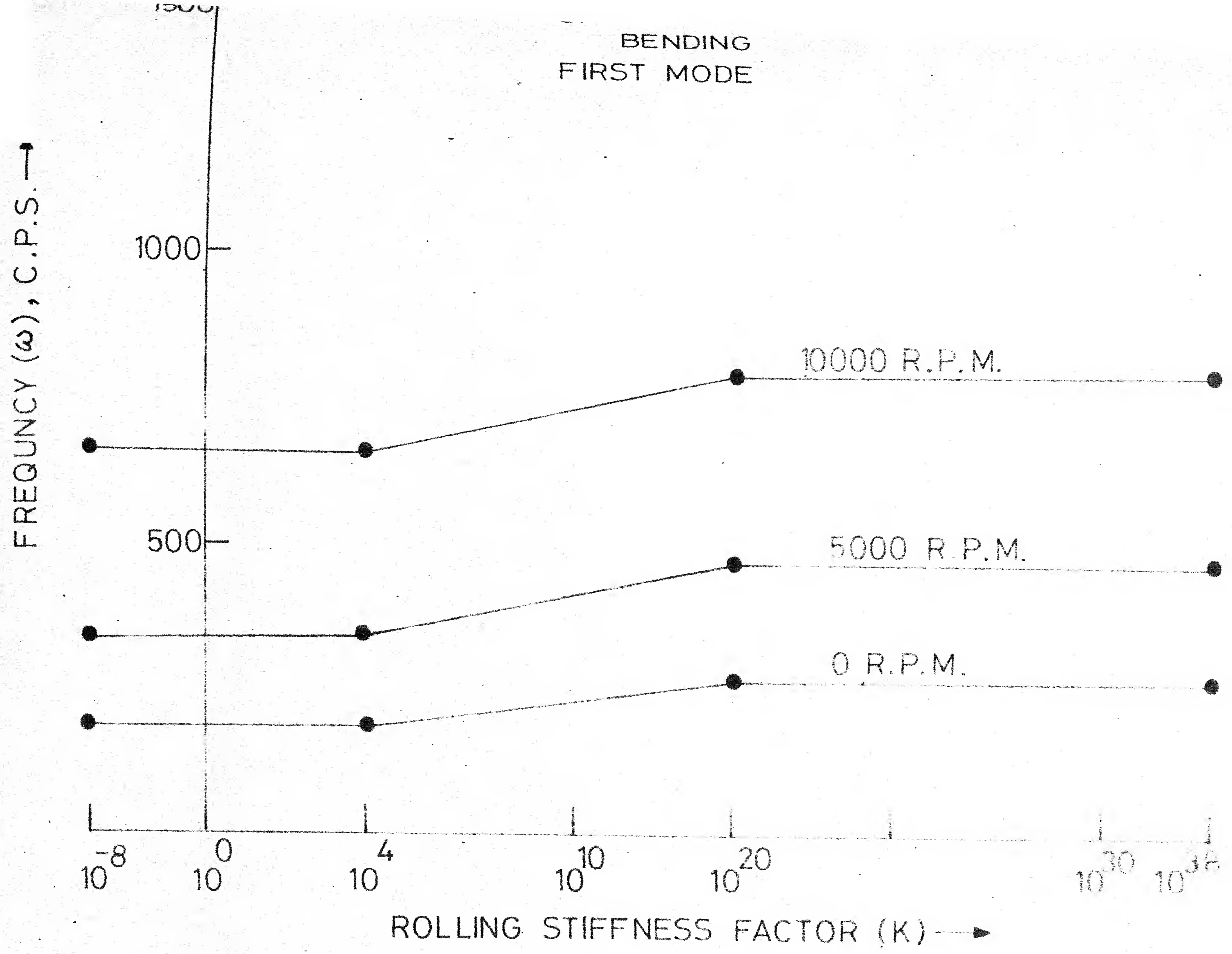


FIG. 5.1.6 - b - EFFECT OF ROOT FLEXIBITY.

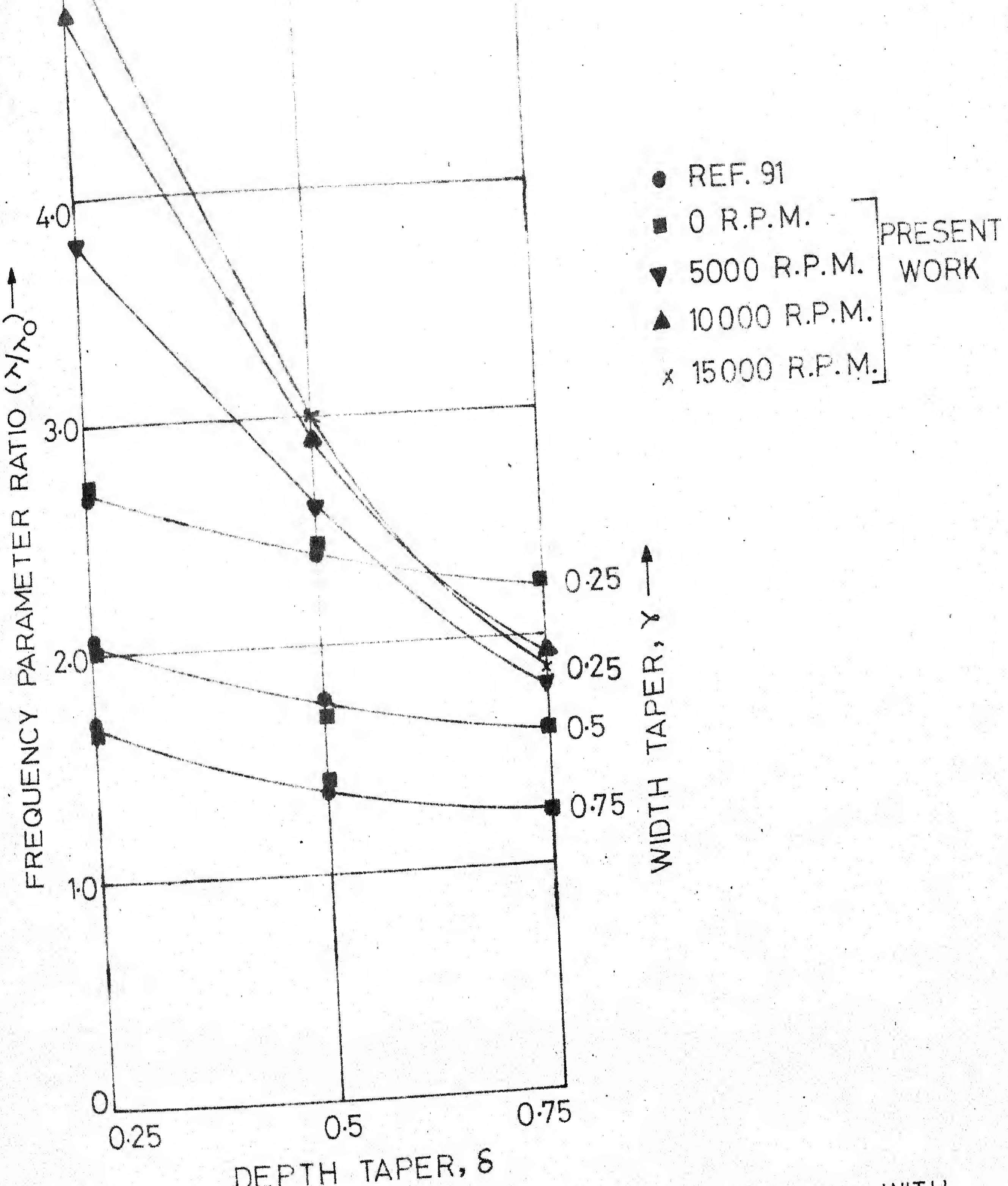


FIG. 5.1·7.a–VARIATION OF FREQUENCY PARAMETER RATIO WITH DEPTH AND WIDTH (FIRST MODE).

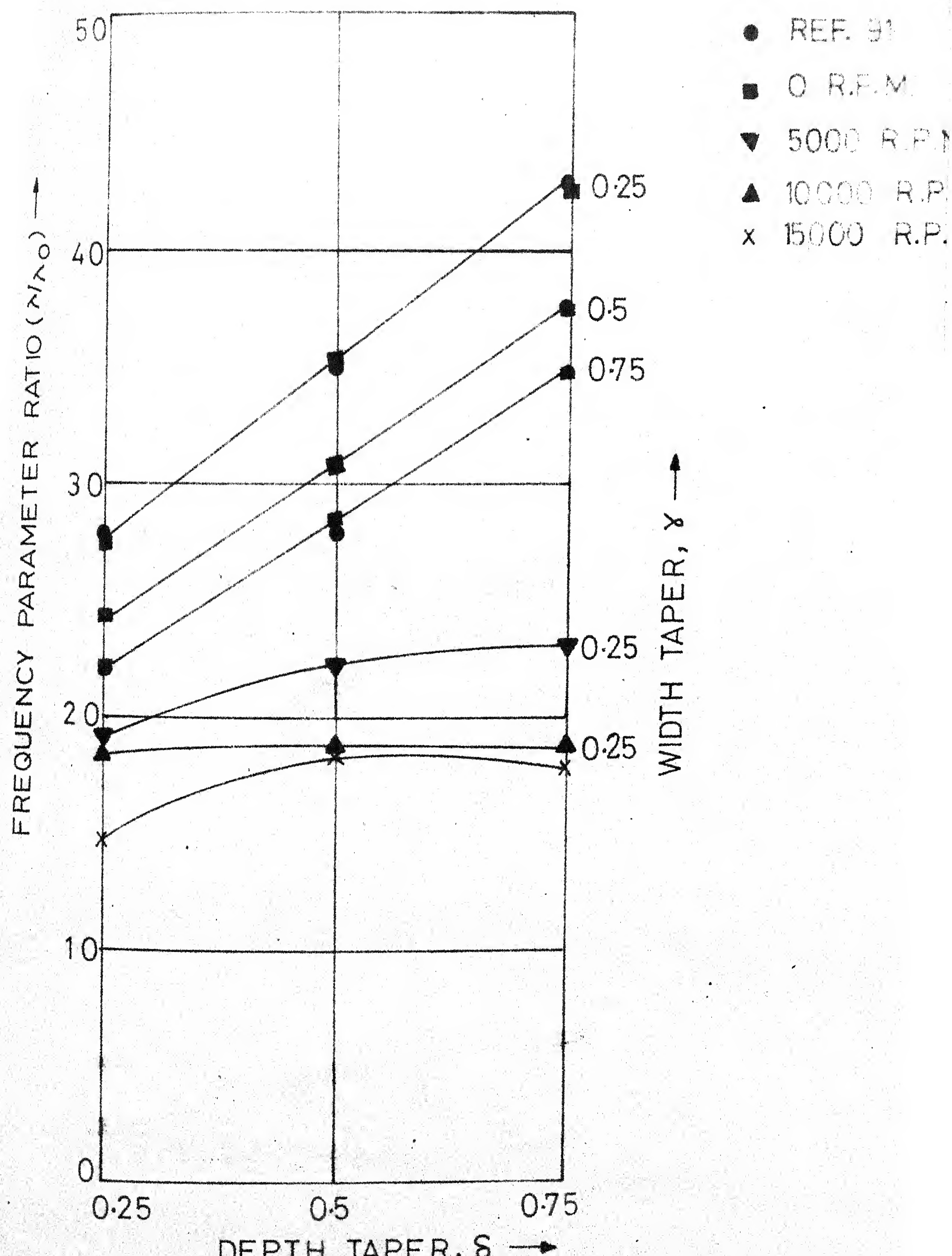


FIG.5.17.b–VARIATION OF FREQUENCY PARAMETER RATIO
DEPTH AND WIDTH (SECOND MODE).

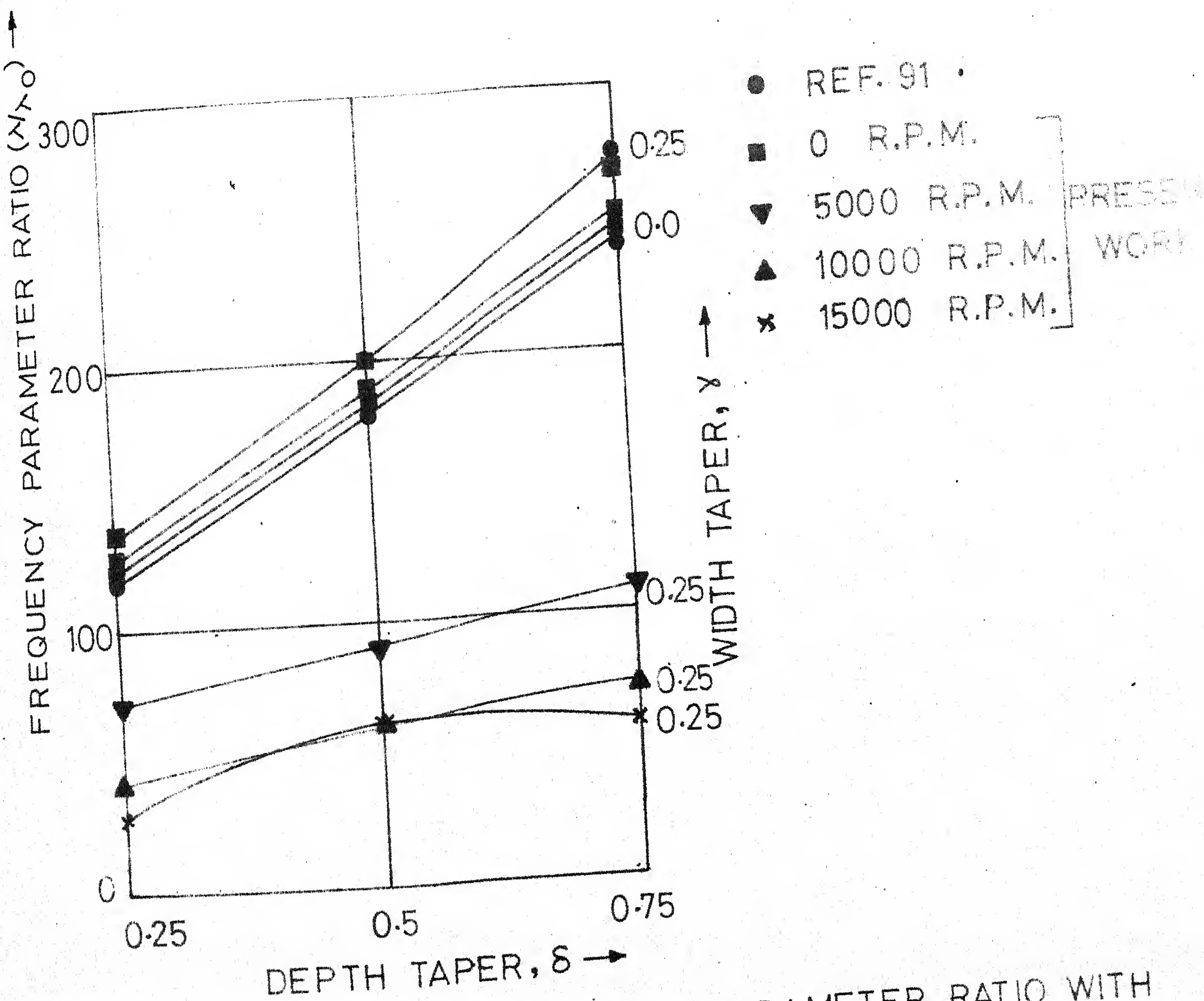
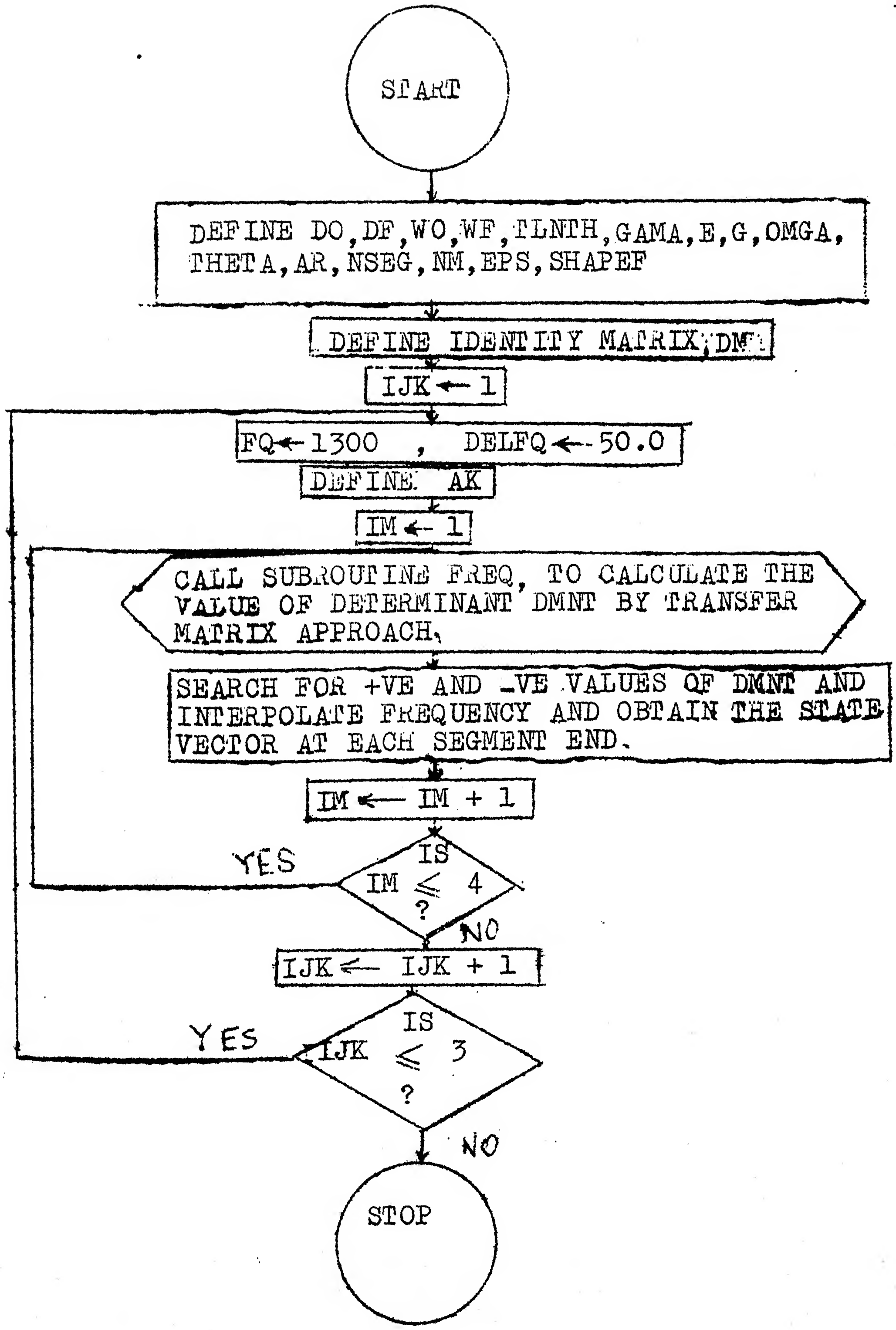
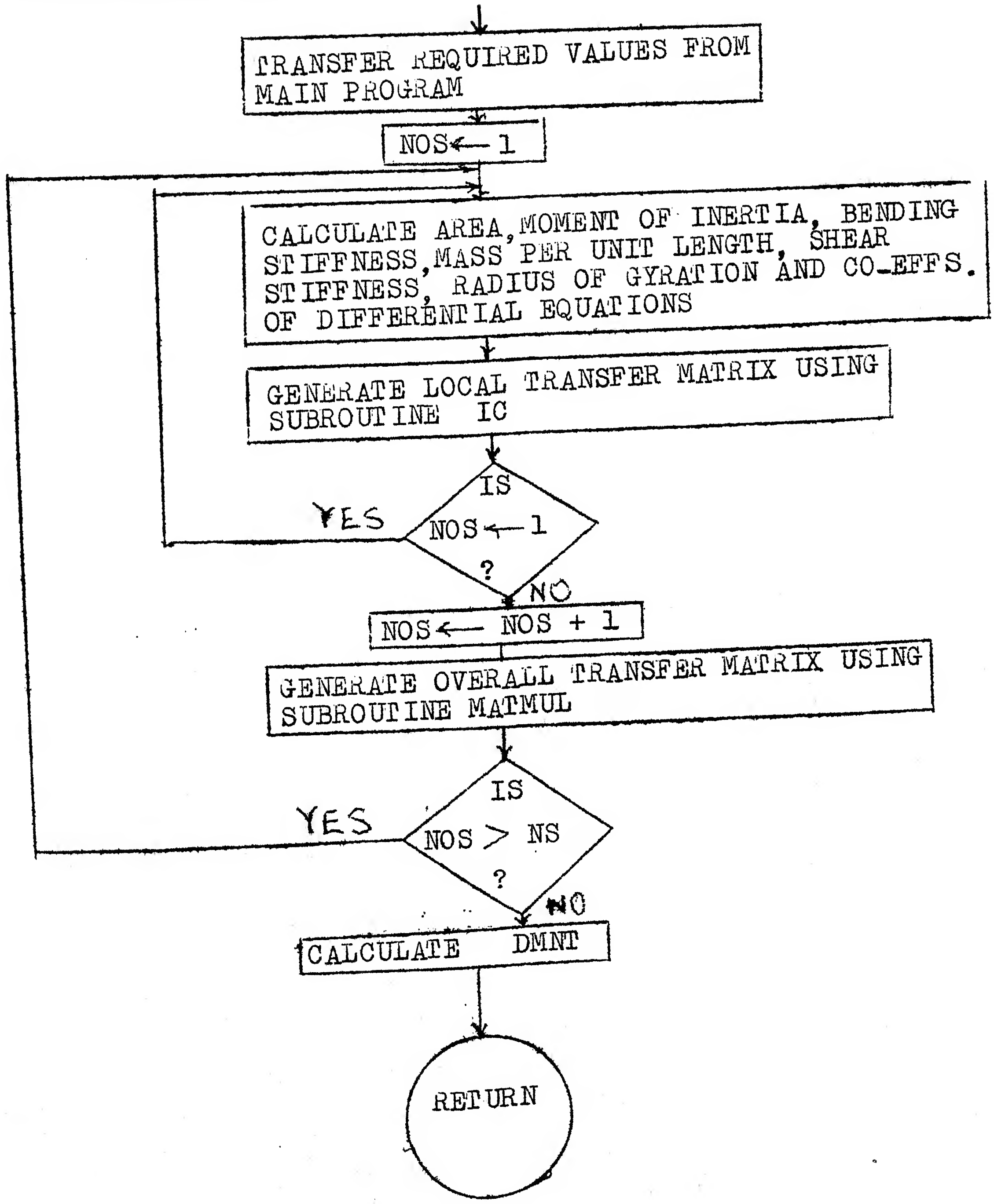


FIG. 5.17c - VARIATION OF FREQUENCY PARAMETER RATIO WITH DEPTH AND WIDTH. (THIRD MODE)

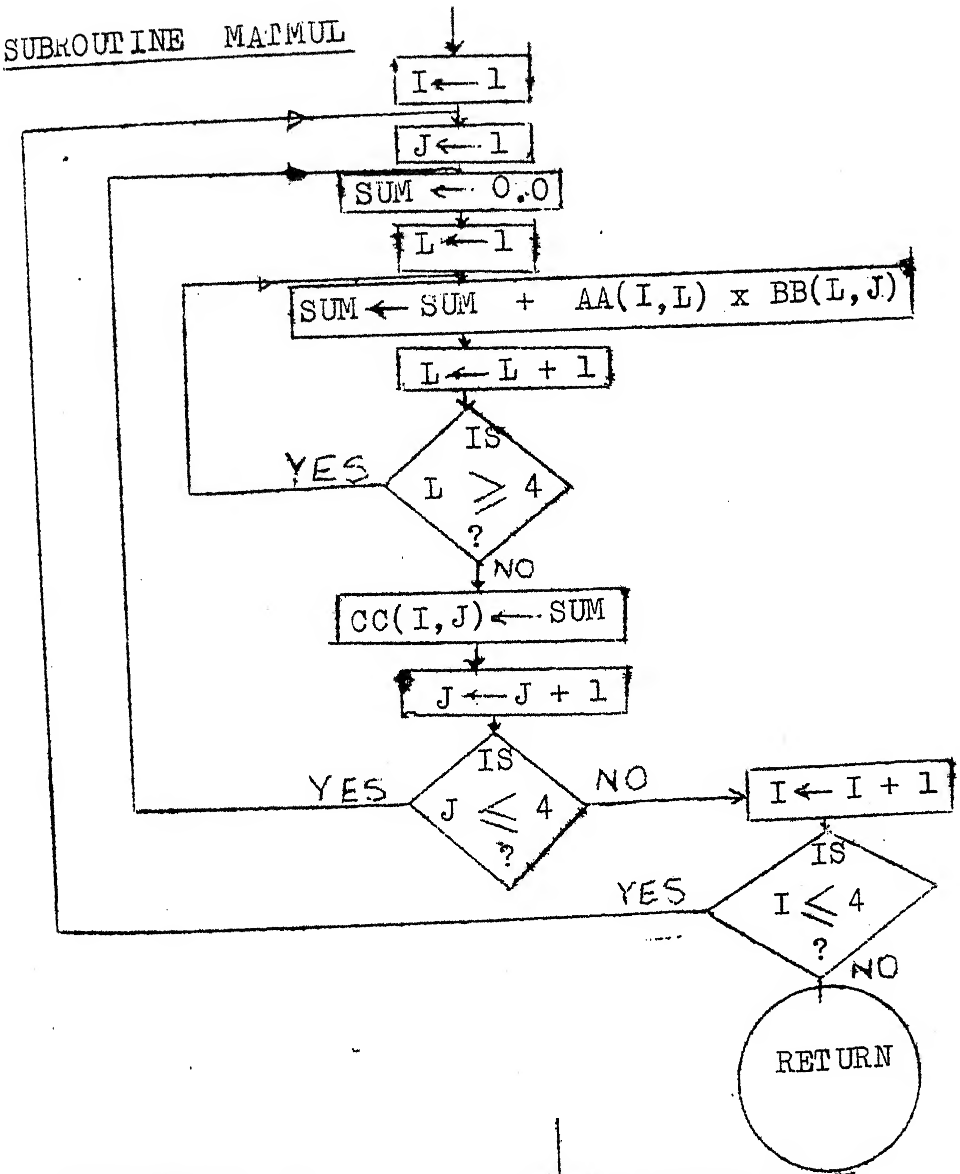


APPENDIX A : FLOW CHART FOR NON-UNIFORM BEAM WITH ROOT FLEXIBILITY

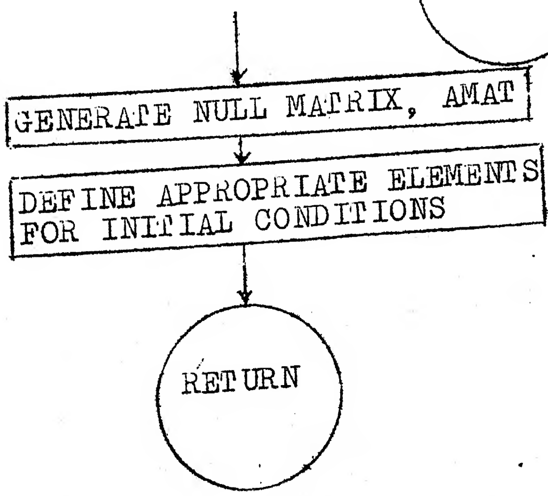
SUBROUTINE FREQ



SUBROUTINE MATMUL



SUBROUTINE IC



B I B L I O G R A P H Y

1. Carnegie, W., ''Solution of Equations of Motion for the Flexural Vibration of Cantilever Blades under Rotation by the Extended Holzer Method'', Bull. Mech. Engng. Educ., p 225 (1967)

2. Duncan, W.J., ''Torsion and Torsional Oscillations of Blades''; Trans. NEC Inst. Engr. and Shipbuilders, 54, p 301 (1938)

3. Duncan, W.J. and Collar, A.R., ''Solution of Oscillation Problems by Matrices'', Phil. Mag., 17, p 866 (1934)

4. Dokumaci, E., ''Development and Application of the Finite Element Method to the Vibration of Beams'', Ph.D Thesis, Univ. Survey, England (1968)

5. Carnegie, W. ; Thomas, J. ; and Dokumaci, E., ''An Improved Method of Matrix Displacement Analysis in Vibration Problems'', Aero. Q., 20, p 321 (1969)

6. Scanlan, R.H. and Rosenbaum, R.,''Introduction to the Study of Aircraft Vibration and Flutter'', MacMillan Co., (1951)

7. Prohl, M.A.,''A General Method of Calculating Critical Speeds of Flexible Rotors'', J.Appl. Mech.,Trans. ASME, p A - 142 (1945)

8. Rao, J.S., ''Vibration of Cantilever Beams in Torsion'', JSER (2), p 351 (1964)

9. Idam, ''A Tabular Procedure for the Determination of Uncoupled Bending Frequency of a Cantilever Beam'', JSER, 10(2), p 187 (1966)

10. Lee, H.C. and Bishop, R.E.D., ''Applications of Integral Equations to the Flexural Vibrations of a Wedge with Rotary Inertia and Shear'', J.Franklin Inst., 277, p 327 (1964)

11. White, W.T., ''An Integral Equation Approach to Problems of Vibrating Beams'', J. Franklin Inst., 245, p 25 and 117, (1948)

12. Martin, A.I., ''Some Integrals Relating to Vibration of Cantilever Beams and Approximation for the Effect of Taper on Overtone Frequencies'', Aero. Q., 7, p 109(1956)

13. Idem, ''Approximation for the Effect of Twist on Vibration of a Cantilever Blade'', ibid, 8, p 291, (1957)

14. Rubin, S., ''Transmission Matrices for Vibration and Their Relation to Admittance and Impedence'', J.Engng. Indus., Trans. ASME, 86, p 9 (1964)

15. Idem, ''Review of Mechanical Immittance and Transmission Matrix Concepts'', J.Accos. Soc. Am. p 1171 (1967)

16. Pestel, E.C. and Leckie, F.A., ''Matrix Method in Elasto-Mechanics'', McGraw Hill Book Co. Inc. (1963)

17. Myklestad, N.O., ''A New Method of Calculating Natural Frequencies of Uncoupled Bending Vibration of Airplane Wings and Other Types of Beams'', J.Aero.Sci., 11, n 4, (April 1944)

18. Duncan, W.J., ''A Critical Examination of the Representation of Massive and Elastic Bodies by Systems of Rigid Masses Elastically Connected'', The Q.J. Mechs. and Appld. Maths., 5, (1952)

19. Lindberg, G.M., ''Vibration of Nonuniform Beams'', Aero. Q., 14, p 387 (1963)

20. Jawson, M.J. and Ponter, A.R., ''An Integral Equation Solution of the Torsional Problem'', Proc. Royal Soc., (1963)

21. Walker, P.B., ''Simple Formulae for Fundamental Natural Freque ncies of Cantilevers'', R and M 1831,(1938)

22. Houbolt, J.C. and Brooks, G.,''Differential Equations of Motion for Combined Flapwise Bending Chordwise Bending and Torsion of Twisted Nonuniform Rotor Blades'' NACA TN-1522 (1948)

23. Whittmeyer, H.,''A Simple Method for the Approximate Computation of All Natural Torsional Frequencies of a Member with Variable Cross Section'', KTH - Aero. TN 13, Stockholm, Sweden (1951)

24. Idem, ''Natural Torsional Frequencies of Bars with Variable Cross Section'', Ingr. Archiv., 20, p 331 (1952)

25. Pipes, A.,''The Matrix Theory of Torsional Oscillations'', J.Appld. Mechs. Trans. ASME,13, p 434 (1942)

26. Thomson, W.T., ''Matrix Solution of Vibration of Nonuniform Beams'', Paper No.49 A -11, presented to ASME (1949)

27. Vet, M.,''Torsional Vibration of Beams Having Rectangular Cross Sections'', J.Acoust. Soc. Ame.,34, p 1570 (1962)

28. Garland, C.F., ''The Normal Modes and Vibrations of Beams Having Noncollinear Elastic and Mass Axes'', J.A.M., Trans. ASME, p A -97 (1940)

29. Isakson, G. and Eisley, G.J., ''Natural Frequencies in Coupled Bending and Torsion of Twisted Rotating and Nonrotating Blades'', NASA No. NSG - 27 - 59 (1964)

30. Rao, J.S,''Coupled Bending Torsion Vibrations of Cantilever and Machines, 3, p 199 (1969)

31. Dunholter, R.J., ''Static Displacement and Coupled Frequencies of a Twisted Bar'', J.Aero. Sci. 13, p 214 (1946)

32. White, W.T., ''An Integral Equation Approach to Problems of Vibrating Beams'', J.Franklin Inst. 245, pp 25 and 117 (1948)

33. Analytical and Experimental Investigations of Effect of Twist on Vibration of Cantilevers, NACA TN-2300(1951) Mendelson, A. and Gendler,S.

34. Diprima,R.C. and Handelman, G.H., ''Vibration of Twisted Bemas'', Q. Appl. Math. 12, p 241 (1954)

35. The Bending Vibrations of a Twisted Rotating Beam, Proc. 3rd Midwestern Conf. Solid Mech., p 177 (1957) - Targoff, W.P.

36. Vibrations of Pretwisted Cantilever Blading, Proc. Inst. Mech. Engr. 173, p 343 (1959) - Carnegie, W.

37. Dawson, B. and Carnegie, W., ''Modal Curves of Pretwisted Beams of Rectangular Cross Section'', ibid, 11, p 1 (1969)

38. ''Flexural Vibration of Pretwisted Tapered Cantilever Beams Treated by Galerkin Method'', J.Engr. Indus., Trans. ASME 94, p 343 (1972) - Rao, J.S.

39. Weidenhammer, F.,''Coupled Bending Vibrations of Turbine Blades in a Centrifugal Force Field'', (in German), Ingenieur Archiv, 39, 5, p. 281 (1970)

40. ''The Coupled Bending Bending Vibration of Pretwisted Tapered Blading'', J.Engr. Indus. Trans. ASME 94, p 255 (1972)- Carnegie, W. and Thomas, J.

41. Idem, ''The Effect of Shear Deformation and Rotary Inertia on the Lateral Frequencies of Cantilever Beams in Bending'', ibid, p 267 (1972)

42. Chen, C., Effect of Initial Twist on Torsional Rigidities, Proc. 1st US Natl. Cong. Appl. Mech., ASME, p 265 (1952)

43. Bogdonoff, J.L. and Horner, J.T., ''Torsional Vibrations of Rotating Twisted Bars'', J.Aero. Sci. 23, p 393 (1956)

44. Brody, W.G. and Targoff, W.P., ''Uncoupled Torsional Vibrations of Rotating Twisted Bars'', WADC, TR 56-501 (1957)

45. Vibrations of Pretwisted Cantilever Blading : An Additional Effect Due to Torsion, ibid, 176, p 315 (1962)- Carnegie, W.

46. Rao, JS 'Torsional Vibration of Pretwisted Cantilever Beams'', J.Inst. Engr. Indus., CE Div. 52, p 211 (1972)

47. Houbolt, J.C. and Brooks, G., Differential Equations of Motion for Combined Flapwise Bending Chordwise Bending and Torsion of Twisted Nonuniform Rotor Blades, NACA 1346 (1958)

48. Carnegie, W. ; Dawson, B. ; and Thomas, J., Vibration Characteristics of Cantilever Blading, Proc. Inst. Mech. Engr. 180 (31), p 71 (1965-66)

49. Dawson, B.Vibration Characteristics of Cantilever Beams of Uniform Cross Section, PhD Thesis, Univ. London (1967)

50. Carnegie, W. and Dawson, B., ''Vibration Characteristics of Straight Blades of Asymmetrical Airfoil Cross Section''; Aero. Q. 20, p 178 (1969)

51. ''Solution of the Equations of Motion of Coupled Bending Bending Torsion Vibrations of Turbine Blades by the Method of Ritz-Galerkin'', Intl. J.Mech. Sci. 12, p 875 (1970)- Rao, J.S. and Carnegie, W.

52. Timoshenko, S.P., ''On the Correction for Shear of the Differential Equation for Transverse Vibrations of Prismatic Bars'', Phil. Mag. 41, p 744 (1921)

53. [illegible], ''A Note on the Use of Variational Method to Derive the Equations of Motion of a Vibrating Beam of Uniform Cross Section Taking into Account of Shear Deflection and Rotary Inertia'', BMEE 2, p 35 (1963) Carnegie, W.

54. Sutherland, R.L. and Goodman, L.E., Vibration of Prismatic Bars Including Rotary Inertia and Shear Deflections, Univ. of III (1954)

55. Chaplin, R., ''Natural Frequencies of Prismatic Blades with Reference to a 12- Stage Turbine'', ARCCP No. 95 (1952)

56. Hurty, W.C. and Rubenstein, M.F., ''Rotary Inertia and Shear in Beam Vibration'', J.Franklin Inst. 278, p 124 (1964)

57. Lee, H.C. and Bishop, R.E.D., ''Applications of Integral Equations to the Flexural Vibrations of a Wedge with Rotary Inertia and Shear'', J. Franklin Inst. 277, p 327 (1964)

58. ''Vibrations of Pretwisted Cantilever Blading Allowing for Rotary Inertia and Shear'', JMES 6, p 105 (1964) - Carnegie, W.

59. Idem, ''Rotary Inertia and Shear Treated by Ritz Method'', J.Aero. Soc. (London) 72, p 341 (1968) - Dawson, B.

60. Rao, J. S 'Effects of Shear and Rotary Inertia on Flexural Vibration of Tapered Beams'', JSER (to appear)

61. Krupka, R.M. and Baumanis, A.M. ''Bending Bending Mode of a Rotating Tapered Twisted Turbomachine Blade Including Rotary Inertia and Shear Deflection'', J. Engr. for Indus., ASME 91, p 1017 (1969)

62. Dawson, B. ; Ghosh, N.G. and Carnegie, W., Modal Curves of Pretwisted Beams Allowing for Shear Deformation and Rotary Inertia, 15th Cong. ISTAM held in Sindri (1970)

63. Jacobsen, L.S., ''Natural Frequencies of Uniform Cantilever Beams of Symmetrical Cross Section'', J. Appl. Mech., ASME 5, p 1 (1938)

64. Idem, ''Natural Periods of Cantilever Beams'', Trans. ASCE 104, p 402 (1939)

65. Scanlan, R.H., ''A Note on Transverse Bending of Beams Having Both Translatory and Rotating Elements'', J. Aero. Sci., 15, p 425 (1948)

66. Kruszewski, E.T., Effect of Transverse Shear and Rotary Inertia on the Natural Frequency of a Uniform Beam, NACA TN 1909 (1949)

67. Fettis, H.E. ''Effect of Rotary Inertia on Higher Modes of Vibration''. ibid, 16. p 445 (1949)

68. Idem, ''The Calculation of Coupled Modes of Vibration by Stodola Method'', ibid, p 259 (1949)

69. Mindlin, R.D. and Deresiewicz, H., Timoshenko's Shear Coefficient for Flexural Vibrations of Beams, Proc. 2nd US Natl. Cong. Appl. Mech., ASME, p 175 (1955)

70. Cowper, G.R., ''The Shear Coefficient in Timoshenko's Beam Theory'', Trans. ASME 88 (series E), p 335 (1966)

71. Gaines, J.H. and Volterra, E., ''Transverse Vibrations of Cantilever Bars of Variable Cross Section'', J. Acoust. Soc. Amer. 39, p 674 (1966)

72. Meyer, F., ''Mathematische Theorie der Transversalen Schwingungen Eines Stabes von Veranderlichen Querschnitt'', Ann. der Physik und Chemie, 33, p 661 (1888)

73. Ward, P.F., ''Transverse Vibrations of a Rod of Variable Cross Section'', Phil. Mag. 25, p 85 (1913)

74. Nicholson, J.W., The Lateral Vibrations of Bars of Variable Cross Section, Proc. Roy. Soc. XCIIIA, p 506 (1917)

75. Wrinch, D.M., Lateral Vibrations of Bars of Conical Type, Proc. Roy. Soc. 101, p 493 (1922)

76. Akimasa, O. ''Lateral Vibrations of Tapered Bars'', JSME 28, p 429 (1925)

77. Conway, H.D., ''Calculation of Frequencies of Truncated Pyramids'', Aircraft Engr. 18, p 235 (1946)

78. Watanabe, I., ''Natural Frequencies of Flexural Vibrations of Tapered Cantilevers with Uniform Thickness'', Proc. 1st Jap. Natl. Cong. Appl. Mech., p 547 (1951)

79. Nabuo, I., ''A Technique for Computing Natural Frequencies of Trapezoidal Vibrators'', J. Aero. Sci. 24, p 239 (1957)

80. Wang, H.C., A General Treatment of the Transverse Vibration of Tapered Beams, PHD Thesis, Univ. III.(1966)

81. Southwell, R.V., ''On a Graphical Method of Determining the Frequency of Lateral Vibration for a Rod of Nonuniform Cross Section'', Phil. Mag. 41, p 419 (1921)

82. Leckie, F.A. and Lindberg, G.M., ''The Effect of Lumped Parameters on Beam Frequencies'', Aero. Q. 14, p 224 (1963)

83. Lindberg, G.M., ''Vibration of Nonuniform Beams'', Aero. Q., 14, p 387 (1963).

84. Martin, A.I., ''Some Integrals Relating to Vibration of Cantilever Beams and Approximation for the Effect of Taper on Overtone Frequencies'', Aero. Q. 7 p 109 (1956)

85. Taylor, J.L., ''Natural Vibrational Frequencies of Flexible Rotor Blades'', Aircraft Engr. 30, p 331(1958)

86. Housner, G.W. and Keightly, W.O., ''Vibration of Linearly Tapered Cantilever Beams'', Proc. ASCE 88 (EM2), p 95 (1962)

87. [illegible], Effect of Taper on Uncoupled Natural Frequencies of Cantilever Beams, PhD Thesis I.I.T., Kharagpur, India (1965) - Rao, J.S.

88. Rissone. R.F. and Williams, J.J., ''Vibration of Nonuniform Cantilever Blades'', ibid, 220, p 497 (1965)

89. Thomas, J., Vibrational Characteristics of Tapered Cantilever Beams, PhD Thesis, Univ. London, England (1968)

90. Carnegie, W. and Thomas, J., ''Natural Frequencies of Long Tapered Cantilevers'', Aero. Q. 18, p 309 (1967)

91. [illegible], ''Determination of the Frequencies of Lateral Vibrations of Tapered Cantilever Beams by the Use of Ritz-Galerkin Process, BMEE 10, p 239 (1971) - Rao, J.S. and Carnegie, W.

92. Thomas, J. and Carnegie, W., Torsional Vibration of Tapered Beams, Proc. 15th Cong. ISTAM, p 29 (1970)

93. [illegible], ''The Effect of Depth Taper on Torsional Vibration of Tapered Cantilever Beams'', JSER 14, p 55 (1970)-Rao,J.S.

94. Mabie, H.H. and Rogers, C.B.,''Transverse Vibrations of Doubly Tapered Cantilever Beams,'' J.Accoust. Soc. Ame., 51, 5, p. 1771. (May 1972)

95. Smith, J.E., ''The Vibration of Blades in Axial Turbomachinary'', Am. Soc. mech.Engrs.,Paper No.66-WA/GT, 12, p.1 (1966)

96. Lo, H. and Renebarger, J., ''Bending Vibrations of a Rotating Beam'', Proc. 1st US Natl. Cong. Appl.Mech.,ASME, P.75(1952)

97. Lo, H. ''A Nonlinear Problem in the Bending Vibration of a Rotating Beam'' J.Appl. Mech., Trans. ASME 19, p. 461 (1952)

98. Isakson, G. and Eisley, G.J., ''Natural Frequencies in Coupled Bending andTorsion of Twisted Rotating and Nonrotating Blades, ''NASA No. NSG-27-59(1964)

99. Boyce, W.E. ; Diprima, R.C. ; and Handleman, G.H., ''Vibrations of Rotating Beams of Constant Section ; Proc. 2nd US Natl. Cong. Appl. Mech., ASME p. 165 (1954)

100.Yntema, R.T., ''Simplified Procedures and Charts for the Rapid Estimation of Bending Frequencies of Rotating Beams'', NACA TN 3459 (1955)

101.Boyce, W.E., ''Effect of Hub Radius on the Vibrations of a Uniform Bar'', J.Appl. Mech., Trans ASME 23, p.287 (1956)

102.Carnegie, W. ''Vibrations of Rotating Cantilever Blading: Theoretical Approaches to the Frequency Problem Based on Energy Methods'', JMES 1, p.235 (1959)

103.Schilhansl, M.J., ''Bending Frequency of a Rotating Cantilever Beam'', J. Appl. Mech., ASME, 25, p. 28 (1958)

104.Carnegie, W. ; Sterling, C. ; and Fleming, J., Vibration Characteristics of Turbine Blading under Rotation- Results of an Initial Investigation and Details of a High-Speed Test Installation, Paper No. 32, Appl. Mech. Convention held at Cambridge, Mass. (1966)

105.Kissel, W., ''The Lowest Natural Bending Frequency of a Rotating Blade of Uniform Cross section'', Eascher Wyss. News 31, p 28 (1958)

106. Rao, J.S.,''Flexural Vibration of Turbine Blades'', Archiwum Budowy Masayn, Tom XVII(3), p 375 (1970)

107. ''Flexural Vibration of Rotating Cantilever Beams'', J.Aero. Soc. India 22, p 257 (1970)

108. Rao, J.S. and Carnegie, W.,''Non-linear Vibration of Rotating Cantilever Beams'', J.Roy. Aero. Soc. 74, p 161 (1970)

109. Idem,''Non-linear Vibrations of Rotating Cantilever Beams Treated by Ritz Averaging Procedure'', J.Roy. Aero.Soc. 76, p 566 (1972)

110. Carnegie, W.,''The Application of the Variational Method to Derive the Equations of Motion of Vibrating Cantilever Blading under Rotation'', Bull.Mech.Engg. Educ., 6, p29 (1967)

111. Ansari, K.A.,''Non-linear Vibration of Rotating Blades'', Ph.D.Thesis, Texas Univ. at Arlington(1972)

112. Pnuelli, D.,''Natural Bending Frequency Comparable to Rotational Frequency in Rotating Cantilever Beam'', J.A.M., 2, p 602, June 1972.

113. Kundu, B.B.''Transverse Vibration of an Isotropic Rotating Beam of Variable Cross-section, ''Indian J.Pure and Appl. Phys., 5, p 247, (1966)

114. Wauer,J.''The Influence of Centrifugal Force Divergence on the Natural Vibration of Rotating Turbine Blades (in german), Forschung im Ingenieurwesan, 39, p 191, (1973)

115. Trompette, P., ''Vibration Analysis of Rotating Turbine Blades'', Paper No. 74-WA/DE-23 for meet Nov. 17-22,1974.

116. Henry, R. and Lalanne, M, ''Vibrational Analysis of Rotating Compressor Blades''; J.Engg. Indus.,Trans. ASME, 2, B, (Aug.1974)

117. Ansari, K.A.,''Nonlinear Vibrations of a Rotating Pretwisted Blade'', Comput. Struct., 5, n 2-3, p 101, (Jun 1975)

118. Liebers, F.,''Resonance Vibrations of Aircraft Propellers'', NACA TM 657 (1936)

119. Idem,''Analysis of Three Lowest Bending Frequencies of a Rotating Propeller'', NACA TM 783 (1936)

120. Love, E.R. and Silberstein, J.P.O.,''Vibration of Stationary and Rotating Propellors'', Rept. SM.48, Div. Aero., Council, Sci. Indus. Res.,Australia,(1946)

121. Billington, A.E.;'The Vibration of Stationary and Rotating Cantilevers with Special Reference to Turbine Blades'', Rept. SM 109 and E.61, Div. Aero., Council, Sci.Indus. Res.,Australia(1948)

122. Sutherland, R.L.,''Bending Vibrations of a Rotating Blade Vibrating in the Plane of Rotation'', J.A.M., ASME, 16, p 389 (1949)

123. Kobari,Y.,''Experiments on Vibrations of Rotating Blades Having Cross-sections of Several Kinds, Proc.1st Jap. Natl.Cong.Appl.Mech.,p 547 (1951)

124. Plunkett, R.,''Free and Forced Vibration of Rotating Blades'', J.Aero.Sci., 18, p 278 (1951)

125. Liner, H.S.,''The Natural Frequencies and Modes of Vibration of a Rotating Beam'', J.Aero.Sci.58, p 552(1954)

126. Giansente, N.,''Rapid Estimation of the Effects of Material Properties on Blade Frequencies'', J.Amer. Helicopter Soc. 16, p 26 (1971)

127. Niordson, F., ''Vibration of Turbine Blades with Loose Hinge Support'', Acta Ployt., Mech. Engg. Series, 3, (1954)

128. Hirsch, G.,''Investigation of Vibrations in Bending of Rotating Turbine Blades on Nonrigid Support'', Jahrbuck Wissonschaft.,Gessellsch Luftfahrt, p 174(1958)

129. Traupel, W.,''Thermische Turbomaschinen'', 2, Springer, Berlin, (1960)

130. Baur, P.''Natural Frequencies of Unshrouded Cantilever Blades'', The Engineer, 218, (Oct. 30, 1964)

131. Horway, G., ''Chordwise and Beamwise Bending Frequencies of Hinged Rotor Blade'', J.Aero. Sci. 15, p 497 (1948)

132. Bogdanhoff, G.,''Linear Vibration of Pinned Rotating Blades'', J.Aerospace Sci.,28 , p. 593 (1961)

133. Goatham, J.I. and Smailes, G.T.,''Some Vibration Characteristics of Pin-Fixed Compressor Blades'', Paper No. 66-WA/GT - 4, ASME, (1966)

134. Shaw, H., ''An Improved Blade Root Design for Axial Flow Compressor (and Turbines)'', J. Aeronautical, 74, p 589 (1970)

135. Ellington, J.P. and McCallion, H.,''Blade Frequency of Turbine-Effect of Disc Elasticity'', Engineering, 187, (1959)

136. Sohngen, H.,''Vibrational Behaviour of Blade Ring in Vacuum'', Dtsch. Versuchsanstalt Luftfahrt, Rept. 1, (1955)

137. Fillipov, A.P.,''Tangential Vibrations of Turbine Blades Taking into Account Vibrations of Disc in Its Plane'', (in Ukranian), Prikl Metkh, 6, n 3, (1960)

138. Wagner, J.T.,''Coupling of Turbomachine Blade Vibrations Through the Rotor'', ASME Paper No.66-WA/GT-5,(1966)

139. Dye, R.C.F. and Henry, T.A.,''Vibration Amplitudes of Compressor Blades Resulting from Scatter in Blade Natural Frequencies'', J.Engng. Power, Trans. ASME, AX,(1969)

140. Shen, F.A.,''Transient Flexible Rotor Dynamics Analysis'', J.Engng.Indus.,B,94, p 531,(1972)

141. Lemke, D.G.and Trumpler, P.R.,''On The Dynamic Response of Axially Coupled Turborotors'', J.Engng. Indus., B 94, p 507 (May 1972)

142. Dopkin, J.A.,''The Effect of Disc Flexibility on Rotor Dynamics'', Rutgers Univ.,The State Univ. of New Jersey, (1972)

143. Young, M.I.,''The Influence of Pitch and Twist on Blade Vibrations'', J.Aircraft, 10, p 383 (1973)

144. Thomas, J. and Dokumaci, E., ''Simple Finite Elements for Pre-twisted Blading Vibration'', Aero.Q.25, p 109 (May 1974)

145. Hoff, N.J.,''The Thermal Barrier-Structures'', Trans. Amer. Soc.Mech.Engrs.,77, n.5, p 759 (July 1955)

146. Gachezov, N.A. and Ronzin, V.D., ''Influence of Circumferencial Non-uniformity of Turbine Inlet Temperature Field on Rotor Blade Vibration Stress Level'', Sov Aeronaut, 16, n. 2, p 112 (1973)

147. Gorelkin, N.M.and Bogov, I.A.,''Thermal Stresses in Gas Turbine Rotor Blades'', (in Russian), Energomashinostroenie, No.10, p 4-7, (Oct.1974)

148.. Linask, I.,''Fracture Mechanics Approach to Turbine Airfoil Design'', ASME Paper No. 75-GT-79.

149. NASA,''Aerodynamic Design of Axial Flow Compressors'', SP- 36 (1965)

150. Marble, F.E.,''Propagation of Stall in Compressor Blade Row'', J.Aero. Sci., 22, p 541 (1955)

151. Shannon, J.F.,''Vibration Problems in Gas Turbines, Cantrifugal and Axial Flow Compressors'', R and M 2226, ARC, p 6, (1945)

152. Mendelson, A.,''Aerodyamic Hysterisis as a Factor in Critical Flutter, J.Aero.Sci.,16, (Nov.1949)

153. Idem, ''Effect of Centrifugal Force on Flutter of Uniform Cantilever Beam at Subsonic Speeds with Application to Compressor and Turbine Blades'', NACA TN 1893 (1949)

154. Technical Report, ''Factors That Affect the Operational Reliability of Turboject Engines'', NASA TR R -54,(1960)

155. Ellison, E.G.,''A Review of the Interaction of Creep and Fatigue'', J.Mech.Engng. Sci., 11, n.3, p 318 (1969)

156. Majumdar, S.,''Low-Cycle Fatigue and Creep Analysis of Gas Turbine Engine Components'', J.Aircraft, 12, p. 376 (Apr.1975)

157. Blackwell, B.D., ''Some Investigations in the Field of Blade Engineering'', J.Roy.Aero.Soc., 62, (Sept.1958)

158. Pearson, H., ''The Aerodynamics of Compressor Blade Vibration'', The Engineer, p 473 (Oct. 1953)

159. Armstrong, E.k. and Stevenson, R.E., ''Some Practical Aspects of Compressor Blade Vibrations'', J.Roy. Aero.Soc., 64, (Mar.1960)

160. Sisto, F., ''Stall Flutter in Cascade'', J.Aero. Sci., 20, p 598 (Sept 1953)

161. Parry, J.F.W. and Pearson, H., ''Casecade Blade Flutter and Wake Excitation'', J.Roy. Aero.Soc.,58, p 505 (July 1958)

162. Pearson, H.,''The Aerodynamics of Compressor Blade Vibration'', Proc. 4th Anglo-American Aeronautical Cong. p 127 (1953)

163. Paranjpe, P.A., ''Self Excited Vibration of Turbomachine Blading'', Zamp 15, p 598 (1964)

164. Pigott, R. and Abel, J.M., ''Vibrations and Stability of Turbine Blades At Stall'', J.Engng. Power, 96, p 201 (jul.1974)

165. Snyder, L.E. and Commerford, G.L., ''Supersonic Unstalled Flutter in Fan Rotors ; Analytical and Experimental Results'', J.Engng. Power, 96, p 379 (Oct. 1974)

166. Pisaraiko, G.S. and Ol'shtein, L.E., ''Problems of Aeroelasticity of Turbomachinery Blades'', (in Russian), Probl Prochn, 6, p 3, (Aug. 1974)

167. Mikolajczak, A.A. ; Arnoldi, R.A.;/Snyder L.E. and Stargardter, H.,'' Advances in Fan and Compressor Blade Flutter Analysis and Predictions'', J.Aircraft, 12, n.4, p. 325, (April 1975)

168. Carta, F.O., ''Coupled Blade-Disc-Shroud Flutter Instabilities in Turbojet Engine Rotors'', J.Engng. Power, 89, p.419 (JUL.1967)

169. Naumann, H. and Yeh, H., ''Lift and Pressure Fluctuations of a Cambered Airfoil Under Periodic Gusts and Applications in Turbomachinery'', J.Engng. Power, Trans. ASME, 95, n.1, (Ja. 1973)

170. Sisto, F.O., ''Aeromechanical Response'', AGARD Lect Ser n 72, (1974)

171. Danforth, C.E., ''Distortion Induced Vibration in Fan and Compressor Blading'', J.Aircraft, 12, n.4, p.216, (Apr.1975)

172. Anderson, R.G. ; Irons, B.M. and Zienkiewicz, O.C., ''Vibration and Stability of Plates using Finite Elements'', Intern. J.Solids and Strs., 4, p. 1031 (Oct. 1968)

173. Rawtani, S. and Dokainish, M.A., ''Vibration Analysis of Pretwisted Cantilever Plates'', CASI Trans., 2, p 95 (Sept. 1969)

174. Idem, ''Vibration Analysis of Rotating Cantilever Plates,'' Intern. J.Num. Methods in Engng., 3, p 233 (Apr.1971)

175. Dokainish, M.A., ''A New Approach for Plate Vibration, Combination of Transfer Matrix and Finite Element Technique'', J.Engng. Indus., Trans.ASME, p. 526 (May 1972)

176. Leckie, F.A., ''The Application of Transfer Matrices to Plate Vibrations'', Ingenieur-Archiv, XXXII, p.100(1963)

177. MacBein, J.C., ''Vibratory Behaviour of Twisted Cantilever Plates'', J. Aircraft, 12, n.4, p.343, (Apr.1975)

178. Smith, D.M., ''Vibrations of Turbine Blades in Packets'', Proc.7th Intl. Cong.Appl.Mechs., 3, (1948)

179. Prohl, H.A., ''A Method of Calculating Vibration Frequency and Stress of a Banded Group of Turbine Buckets,'' ASME Paper. 56-A-116, (1956)

180. Tuncel, C., Dueckner, H.F. and Koplik, B., ''An application of Dickoptics in the Determination of Turbine Bucket Frequencies by the use of Perturbations'', ASME Paper No. 69-Viber-57, (1969)

181. Fujino, T. ''A Theoretical Consideration on the Vibration of Turbine Blade System'', Bull.Jap.Soc.Mech.Engng., 1, n.2, (1958)

182. Singh, B.N. and Nandeoswaraiya, N.S., ''Vibration Analysis of Shrouded Turbine Blades'', J.Instn.Engrs., India, 40, n.1, part 2, (Sept.1959)

183. Stargardter, H., ''Dynamic Models of Vibrating Rotor Stages'', ASME Paper No.66-WA/GT-8, (1966)

184. Bhide, V.G., ''Vibrations of Packetted Turbine Blades'', M.Tech. Thesis, Deptt. of Mech.Engg., IIT Kanpur, (Aug.1972)

185. Bajaj, G.R., ''Free Vibration of Packetted Turbine Blades-Coupled Bending Bending Torsion Modes, M.Tech. Thesis, Deptt. of Mech. Engg., IIT Kanpur (July 1974)

186. Ohtsuke, M., ''Untwist of Rotating Blades'', J.Engng. Power, 97, A, 2, p. 180 (Apr.1975)

187. Srinivasan, A.V., ''Non-linear Vibrations of Beams and Plates'', Intl. J.Nonlinear Mechs. 1, p 179 (Nov.1966)

188. Sathyamoorthy, M. and Pandalai, K.A.V., ''Governing Equations for Large Amplitude Flexural Vibrations of Plates and Shells'', IIT(M) AE-R. No.19, (Feb.1972)

189. Idem, ''Large Amplitude Vibration of Deformable Bodies'', Aero.Soc. of India, (Feb.1973)

190. Hanson, M.P., ''Effect of Blade-Root Fit and Lubrication on Vibration Characteristics of Ball Root Type Axial-Flow Compressor Blades'', NACA RM E50C17 (1950)

191. Goodman, L.E. and Klumpp.J.H., ''Analysis of Slip Damping with Reference to Turbine Blade Vibration'', J.A. M., Trans. ASME, 23, n 3, (Sept. 1956)

192. Kozolov, I.A.,''Structural Dissipation of Energy During Vibration of Turbine Blades'', (in Russian) V.Sb., Rassoyanie Energii Pri Kolebamiyakh Uprug. Sistem., (1963)

193. Lazan, B.J. and Goodman, L.E., ''Effect of Material and slip Damping on Resonant Behavior'', Appl. Mechs. Div. Urbana, Illinois, (June 1956)

194. Lazan, B.J., ''Energy Dissipation in Structures with particular Reference to Material Damping in Structural Damping'', (Editor J.E. Razicka), ASME, New York,(DEc. 1959)

195. Morduchow, M.,''On Internal Damping of Rotating Beams'', NACA TN - 1996 (1949)

196. Pisarenko, G.S., ''Vibration of Elastic Systems Taking into Account of Energy Dissipation in the Material, WADD TR. 60- 582, (Feb.1962)

197. Baker, W.E. ; Woolam, W.E.; and Young, D.,''Air and Internal Damping of Thin Cantilever Beams'', Intl. J.Mech. Eng. Sci., 2, (1967)

198. Kerlin, R.L.,''Transverse Vibration of Non- uniform, Internally Damped Cantilever Beams'', J.Acc.Soc.Amer., 50, p 95, (1971)

199. Gluin, J.and Maybee, J.S., ''The Role of Material Damping in the Stability of Rotating Systems'', J.Sound and Vibr., 21, p 399, (Apr. 1972)

200. Pearson, H.,''The Aerodynamics of Compressor Blade Vibration'', Proc. Fourth Anglo-American Aeronautical Conf.,London, (1953)

201. Caville, Y. ; Aureille, R., and Chancelle, H.,''Study of Aerodynamic Damping of Turbomachine Blade Vibration'', (in French), Honille Blanche no.6, p.531-540, (1971)

202. Jones, D.I.G. ; Nashif, A.D. and Stargardter, H.,''Vibrating Beam for Reducing Vibration in Gas Turbine Blade'', J.Engng. Power, Trans. ASME, 97, p 111 (1975)

203. Parin, M.L. and Jones, D.I.G.,''Encapsulated Tuned Dampers for Jet Engine Component Vibration Control'', J.Aircraft, 12, p.293 (Apr. 1975)

204. Simmons, W.F., ''Current and Future Materials Usage in Aircraft Gas Turbine Engines'', Met. and Cerm Inf Cent Rep 73 - 14, p 87 (Jan-1973)

205. El., M., ''New Materials for Aeronautical Turbines'', Gammel Entropie, 9, p 25, (in French), (May 1973)

206. Huff, H., and Betz, W., ''Impact of Composite Material On Aerospace Vechicles and Propulsion Systems,'', AGARD, p 6, (May 1973)

207. Jahnake, L.P. and Brnch, C.A., ''Requirements For and Characteristics Demanded of High Temperature Gas Turbine Components'', AGARD Conf. Proc. n. 156, p 3 - 14 (April 1974)

208. Edres, W., ''High Temperature Mater. in Gas Turbines, Proc. Prop. and Discuss., Baden, Switz, p. 1-14 (Mar 12, 1973), Publ. by Am Elsevier Publg. Co., New York, (1974)

209. Johns, R.H., ''Impact Testing of Composite Fan Blades'', NASA Lewis Res. Cent., Ohio, SAMPE Q, p. 14, (July 1974)

210. Nabatova, N.A. and Shipov, R.A., ''Effect of the Material of the Rotor Blades of Axial Compressors on the Regimes Giving Rise to Flutter'', (in Russian), Probl Prochn, 6, n.8, p.63-67 (Aug. 1974)

211. Visser, C. ; Lien, S. and Schaller, R.J., ''Application of Finite Element Analysis to Ceramic Components'', J. Am. Ceram Soc., 58, n 3 - 4, p.131 -135 (Mar - Apr. 1975)

212. Schaller, R.J. ; Visser, C. and Lien, S., ''Ceramic Rotating Blades : Some Critical Design Parameters For Gas Turbine Applications'', J. Engng. Power, Trans ASME, 97, A, n 3, p.319 - 328, (Jul.1975)

213. Gaytor, H., ''High Temperature Alloys For Gas Turbines'', Aircraft Engineering, 47, n 1, p.14, (Jan.1975)